Free Integral Calculus

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December 4, 2023

<u>Problem</u> $(\mathcal{A}, \varphi)$ ncps

X, Y free $\quad X \sim \mu, \; Y \sim \nu$

$P \in \mathbb{C}\langle X, Y\rangle$

Q: compute distr. of $P(X,Y)$

$\varphi(P(X,Y)^n) \qquad n \in \mathbb{N}$

mgf $\quad M(z) = \varphi\left((1 - zP(X,Y))^{-1}\right)$

Solved for

$P(X,Y) =$	$X+Y$	$\mu \boxplus \nu$	Voiculescu 1986
	$X \cdot Y$	$\mu \boxtimes \nu$	– " – 1987
	$i[X,Y]$	free commutator	Nica/Speicher 1998
	$XY+YX$	anticommutator	Vasilchuk 2003
		FMNS 2020	
		Perales 2021	
	general	linearization	Belinschi/Mai/Speicher 2017

Classical case:

$$\mathbb{E}\, f(X,Y) = \iint f(s,t)\, d\mu(s)\, d\nu(t) \quad \text{double integral}$$

$$= \mathbb{E}\left[\mathbb{E}_Y f(X,Y)\right]$$

Similar approach in free case Voiculescu / Biane 90s

$$E_X\left[(z-(X+Y))^{-1}\right] = \left(\omega_1(z) - X\right)^{-1}$$

$\Rightarrow$ subordination

$$G_{X+Y}(z) = G_X(\omega_1(z))$$

Cond exp: $(\mathcal{A}, \varphi)$, $\mathcal{B} \subseteq \mathcal{A}$ subalg

map $E: \mathcal{A} \to \mathcal{B}$ s.t. $\varphi \circ E$

$$\varphi(ab) = \varphi(E[a]b) \qquad \begin{array}{l}\forall a \in \mathcal{A} \\ \forall b \in \mathcal{B}\end{array}$$

$\to$ $\mathcal{B}$-modul map $E[b_1 a b_2] = b_1 E[a] b_2$

1. does not need to exist
2. does exist if $\mathcal{A}, \mathcal{B}$ vN alg, φ trace
3. exists $(\mathcal{A}, \varphi) = (\mathcal{A}_1, \varphi_1) * (\mathcal{A}_2, \varphi_2)$, $\mathcal{B} = \mathcal{A}_1$

Goal: $E_X[(1 - z P(x, y))^{-1}]$

Main Result

- $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_n$
- $T = \sum C_i \otimes a_i$ a linear matrix pencil ($a_i \in \mathcal{A}_i$, $C_i \in M_n(\mathbf{C})$)
- $I \sqcup J = [n]$ a partition
- $\mathbb{E}_I$ the conditional expectation onto $\mathcal{B} = \bigstar_{i \in I} \mathcal{A}_i$
- $\widetilde{\eta}_i(z) = \sum_{n=1}^{\infty} \beta_n(a_i) z^{n-1}$ the shifted Boolean cumulant generating function of the random variable a_i

Then

$$\mathbb{E}_I\left[\left(I - z \sum C_i \otimes a_i\right)^{-1}\right] = \left(I - z \sum_{i \in I} C_i \otimes a_i - z \sum_{j \in J} C_j F_j\right)^{-1}$$

where $F_1, F_2, \ldots, F_n : \mathbf{C} \to M_n(\mathbf{C})$ constitute the unique analytic fixed point of the system of equations

$$F_i = \widetilde{\eta}_i\left(z\left(I_N - z \sum_{j \neq i} C_j F_j\right)^{-1} C_i\right) \qquad \text{for } i = 1, 2, \ldots, n$$

case of 2 variables: X, Y free

$$E_X\left[\left(I - z\, C_X \otimes X - z\, C_Y \otimes Y\right)^{-1} \right]$$

$$= \left(I - z C_X \otimes X - z\, C_Y \underline{\underline{F_Y}}\right)^{-1}$$

$$= \left(I - z H_Y C_X \otimes X\right)^{-1} \cdot H_Y \qquad H_Y = \left(I - z\, C_Y F_Y\right)^{-1}$$

$$H_X = \left(I - C_X F_X\right)^{-1}$$

equations: $\begin{cases} F_X = \tilde{\eta}_X \left(z H_Y C_X\right) \\ F_Y = \tilde{\eta}_Y \left(z H_X C_Y\right) \end{cases}$ $\quad F_X, F_Y$ analytic $\underline{\text{at } 0}$

$\underline{\text{Corollary}}$: $\left(1 - z^m P(X,Y)\right)^{-1} = \underline{u}^T \left(I - z\left(C_X \otimes X + C_Y \otimes Y\right)\right)^{-1} \underline{v}$

$$E_X\left[\left(1 - z^m P(X,Y)\right)^{-1}\right] = u^T E_X \left(\qquad \right)^{-1} v$$

$m = \deg P(X,Y)$

$$= u^T \left(I - z H_Y C_X \otimes X\right)^{-1} H_Y v$$

$$M_{P(X,Y)}(z^m) = u^T M_X(z H_Y C_X) H_Y v$$

$=$

$\Rightarrow$ eigenvals of $zH_X C_X$ ~ "Subordination function"

main tool Boolean cumulants of free variables

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in I(n)} \beta_\pi(a_1 \dots a_n)$$

recurrence $$\varphi(a_1 \dots a_n) = \sum_{k=1}^{n} \beta_k(a_1, a_2, \dots, a_k)\varphi(a_{k+1} \dots a_n)$$

deconcatenation

$$\bar{\Delta}_d(a_1 \dots a_n) = \sum_{k=1}^{n-1} a_1 \dots a_k \otimes a_{k+1} \dots a_n$$

$$\tilde{\eta}_X(z) = \sum_{n=1}^{\infty} \beta_n(X) z^{n-1} = \frac{1}{z}\eta_X(z)$$

$$M_X(z) = \frac{1}{1 - \eta_X(z)}$$

Prop $\beta_n(a_1, \dots, a_n) = 0$ if $a_i \in \mathcal{A}_1 \cup \mathcal{A}_2$, $\mathcal{A}_1, \mathcal{A}_2$ free s.t. a_1, a_n come from different algebras

Property (CAC) cyclically alternating

Proof: $\beta_n(a_1 \dots a_n) = \sum_{\pi \in NC_{irr}} r_\pi(a_1 \dots a_n)$

Prop if φ is a trace the (CAC) $\iff$ freeness

Prop $\quad \vec{\nabla} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a derivation

$$a \longmapsto 1 \otimes a - a \otimes 1$$

Leibniz: $\vec{\nabla}(a.b) = \vec{\nabla} a \cdot (1 \otimes b) + (a \otimes 1) \vec{\nabla}(b)$

clear $\quad \varphi \otimes \varphi(\vec{\nabla}(a)) = 0$

if $\mathcal{A} = \mathbb{C}\langle X, Y\rangle$

$$\vec{\nabla}_X(x^k) = 1 \otimes x^k - x^k \otimes 1 \qquad \vec{\nabla}_Y$$

$$\vec{\nabla}_X(y^\ell) = 0$$

Th $\quad \vec{\nabla} = \vec{\nabla}_X + \vec{\nabla}_Y$

Property $(\vec{\nabla})$ $\quad \varphi \otimes \varphi(\vec{\nabla}_X(X_1 Y_1 \dots X_n Y_n)) = 0 \qquad X_i = X^{k_i}, \; Y_i = Y^{\ell_i}$

Th (Voiculescu) if φ is a trace

the Property $(\vec{\nabla}) \iff$ freeness

Prop Property $(\nabla) \iff$ (CAC)

$$\varphi(a_1 \dots a_n) = \sum_{h=1}^{n} \beta_h(a_1 \dots a_k)\, \varphi(a_{k+1} \dots a_n)$$

Prop $E_X[Y_0 X_1 Y_1 \dots X_n Y_n] = \sum_{k=0}^{n} \beta_{2k+1}(Y_0, X_1, \dots, X_k Y_k)\, E_X(X_{k+1} \dots X_n Y_n)$

$\mathcal{A} = \mathbb{C}\langle X, Y\rangle$
$\mathcal{B} = \mathbb{C}\langle X\rangle$

Proof find $u \in \mathbb{C}\langle X\rangle$ s.t.

$$\varphi(Y_0 X_1 Y_1 \dots X_n Y_n \cdot Z) = \varphi(u \cdot Z) \quad \forall Z \in \mathbb{C}\langle X\rangle$$

$n=2$

$$\varphi(Y_0 X_1 Y_1 X_2 Y_2 Z) = \beta_1(Y_0)\varphi(X_1 Y_1 X_2 Y_2 Z) + \underbrace{\beta_2(Y_0, X_1)}\varphi(Y_1 X_2 Y_2 Z)$$

$$+ \beta_3(Y_0, X_1, Y_2)\varphi(X_2 Y_2 Z) + \cancel{\overset{=0}{\beta_4}(Y_0, X_1, Y_1, X_2)\varphi \dots)}$$

$$+ \beta_5(Y_0, X_1, Y_2, X_2, Y_2)\varphi(Z) + \cancel{\beta_6(Y_0 \dots Z)}$$

$$= \varphi\Big(\underbrace{\beta_1(Y_0) E_X[X_1 Y_1 X_2 Y_2] + \beta_3(Y_0, X_1, Y_2) E_X[X_2 Y_2] + \beta_5(\dots)}_{E_X[Y_0 X_1 Y_1 X_2 Y_2]}\, Z\Big)$$

Define partial deconcatenation moving X to the right

$$\vec{\Delta}_X Y_0 X_1 Y_1 \dots X_n Y_n = \sum_{k=1}^{n} Y_0 \dots Y_k \otimes X_{k+1} Y_{k+1} \dots X_n Y_n$$

block-beta $\vec{\Delta}_X Y_0 = 0$

$$\beta^b(Y_0 X_1 Y_1 \dots X_n Y_n) = \beta_{2n+1}(Y_0, X_1, Y_1, \dots, X_n, Y_n)$$

Th $E_X[W] = \beta^b(W) + \beta^b \otimes E_X[\vec{\Delta}_X W]$

if $W = Y_0 X_1 \dots X_n Y_n$ start & end in Y

fails for other words

<u>fix</u>

$$\beta_Y^b(W) = \begin{cases} \beta^b(W) & \text{if } W = Y_0 \dots Y_n \\ 0 & \text{otherwise} \end{cases}$$

$\beta_X^b(W)$

(CAC) $\Rightarrow \beta^b = \beta_X^b + \beta_Y^b$

Prop $E_x[w] = \beta_y^b(w) + \beta_y^b \otimes E_x[\vec{\Delta}_x w] \quad \forall w$

$= \qquad\qquad \vec{\nabla}_x$

$\vec{\nabla}_x = \vec{\Delta}_x - \overleftarrow{\Delta}_x$ is a derivation

$\beta_y^b \otimes E_x \overleftarrow{\Delta}_x w = 0$

extra dies annihilated by β_y^b

$= \beta_y^b(w) + \beta_y^b \otimes E_x[\vec{\delta}_x w]$

$\vec{\delta}_x = (1 \otimes x)\partial_x$

↪ free diff quot

$\vec{\delta}_x x^k = \sum_{l=0}^{k} x^l \otimes x^{k-l}$

$= 1 \otimes x^k + x \otimes x^{k-1} + \dots + x^{k-1} \otimes x$

Applicati: $R = (z - P)^{-1}$

$$\breve{\delta}_x R = (R \otimes 1)\, \breve{\delta}_x P \,(1 \otimes R)$$

ex additive subordination

$$P(X,Y) = X + Y \qquad R = (z - X - Y)^{-1}$$

$$\breve{\delta}_x P = 1 \otimes X$$

$$\breve{\delta}_x R = R \otimes XR$$

$$E_X[\underbrace{(z - X - Y)^{-1}}_{R}] = \beta_Y^b(R) + \beta_Y^b \otimes E_X[R \otimes XR]$$

$$= \beta_Y^b(R) + \beta_Y^b(R)\, X E_X[R]$$

$$\Rightarrow E_X[R] = \left(1 - \beta_Y^b(R) X\right)^{-1} \beta_Y^b(R)$$

$$= \left(1/\beta_Y^b(R) - X\right)^{-1}$$

$$\Rightarrow 1/\beta_Y^b(R) = \omega_1(z)$$

$$\beta_Y^b(R) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{\beta_Y^b((X+Y)^n)}{z^n}$$

$\rightarrow$ evaluat $\beta_Y^b(R)$?

ex anti comutator:

$$P = XY + YX$$

$$\psi = (1 - z^2(XY+YX))^{-1}$$

$$\overset{\smile}{\partial}_X P = 1 \otimes XY + Y \otimes X$$

$$\overset{\smile}{\partial}_X \psi = z^2(\psi \otimes XY\psi + \psi Y \otimes X\psi)$$

$$E_X[\psi] = \beta_Y^b(\psi) + z^2\beta_Y^b(\psi)\, X \underbrace{E_X[Y\psi]} + z^2\beta_Y^b(\psi Y)\, X\, E_X[\psi]$$

$$E_x[\Psi\psi] = \beta_\Psi^b(\Psi\psi) + z^2 \beta_\Psi^b(\Psi\psi) \times E_x[\Psi\psi] + z^2 \beta_\Psi^b(\Psi\psi\Psi) \times E_x[\psi]$$

→ two eqs for $E_x[\psi]$ and $E_x[\Psi\psi]$

→ elimination of $E_x[\psi]$