A central limit theorem for tensor products of free random variables

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Joint work with C. Lancien and P. Youssef

Setting

Let $(\mathcal{A}, \tau)$ be a noncommutative probability space. That is, $\mathcal{A}$ is a unital algebra equipped with an involution $(a^*)^* = a$ and a tracial faithful state τ, that is,

1. $\tau(1) = 1$;
2. $\tau(ab) = \tau(ba)$;
3. $\tau(aa^*) \geq 0$ and equality holds iff $a = 0$.

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Examples of such spaces are $(L^\infty(\mathbb{P}), \mathbb{E})$ and $(M_n(\mathbb{C}), \operatorname{tr}/n)$. The distribution of $a = a^* \in \mathcal{A}$ is given by

$$\{\tau(a^n) : n \in \mathbb{N}\}; \quad \tau(a^n) = \int x^n \, \mathrm{d}\mu_a.$$

The convergence in distribution is characterized by the convergence of the moments.

Tensor product

Let $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$ be two noncommutative probability spaces. We equipped the tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ with the trace $\tau_1 \otimes \tau_2$ such that

$$\tau_1 \otimes \tau_2(a_1 \otimes a_2) = \tau_1(a_1)\tau_2(a_2),$$

and extended by linearity. The space $(\mathcal{A}_1 \otimes \mathcal{A}_2, \tau_1 \otimes \tau_2)$ is again a noncommutative probability space.

Freeness

Definition 1

We say that subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_d \subseteq \mathcal{A}$ are free if

$$\tau(a_1 \cdots a_k) = 0,$$

whenever

1. $k \geq 0$;
2. $a_i \in \mathcal{A}_{j_i}$, $\tau(a_i) = 0$, $j_i \in [d]$, for all $i \in [k]$;
3. $j_1 \neq j_2, \ldots, j_{k-1} \neq j_k$.

Random variables $a_1, \ldots, a_d \in \mathcal{A}$ are free whenever their spanning algebras are.

Free central limit theorem

Theorem 1 (Voiculescu - '85)

Let $a, a_1, \ldots, a_n \in \mathcal{A}$ be free i.i.d self-adjoint random variables with mean $\lambda = \tau(a)$ and variance $\sigma^2 = \mathrm{var}(a)$. Then

$$S_n = \frac{1}{\sigma\sqrt{n}} \sum_{k\in[n]} a_k - \lambda$$

converges in distribution to the semi-circle law μ_{sc}, whose density is given by

$$f(x) = \frac{1}{2\pi}\sqrt{4 - x^2}1_{|x|\leq 2}.$$

Model

Let $a, a_1, \ldots, a_n \in \mathcal{A}$ be free i.i.d self-adjoint random variables with mean $\lambda = \tau(a)$ and variance $\sigma^2 = \mathrm{var}(a)$. Let $b_k \in \mathcal{A} \otimes \mathcal{A}$ given by

$$b_k = \frac{a_k \otimes a_k - \tau \otimes \tau(a \otimes a)}{\sqrt{\mathrm{var}(a \otimes a)}}.$$

We are interested in the central limit theorem for b_k, namely, the convergence in distribution of

$$S_n := \frac{1}{\sqrt{n}} \sum_{k \in [n]} b_k. \tag{1}$$

Motivation - Quantum channels

Consider $M_k \in M_d(\mathbb{C})$ self-adjoint i.i.d random matrices. A random quantum channel can be described as

$$M := \frac{1}{\sqrt{n}} \sum_{k \in [n]} M_i \otimes \overline{M}_i - \mathbb{E}\, M_i \otimes \overline{M}_i.$$

Under some conditions [Lancien, S., Youssef - '23], the spectral distribution of M converges as d goes to infinity to

$$S_n = \frac{1}{\sqrt{n}} \sum_{k \in [n]} a_k \otimes a_k - \tau \otimes \tau(a_k \otimes a_k).$$

Model

Let $\delta^2 = \mathrm{var}(b_1) = \sigma^2(\sigma^2 + 2\lambda^2)$. Then, we can rewrite

$$S_n = \frac{1}{\delta\sqrt{n}} \sum_{k\in[n]} a_k \otimes a_k - \lambda^2.$$

Remark 1.1

Even though the variables a_k are free, the variables $a_k \otimes a_k$ are not usually free. This was proved by Collins-Lamarre 2016. So, the usual free central limit theorem is not applicable.

Result

Theorem 2 (L. S. Y - '24)

Let $a, a_1, \ldots, a_n \in \mathcal{A}$ be free i.i.d self-adjoint random variables with mean $\lambda = \tau(a)$ and variance $\sigma^2 = \mathrm{var}(a)$. Let

$$q = \frac{2\lambda^2}{\sigma^2 + 2\lambda^2} \in [0, 1].$$

Then the normalized sums S_n for b_k converges in distribution to

$$\mu_q := \sqrt{q}\left(\frac{1}{\sqrt{2}}\mu_{sc} + \frac{1}{\sqrt{2}}\mu_{sc}\right) \boxplus \sqrt{1-q}\,\mu_{sc}.$$

Scheme

Theorem 3 (Bozejko-Speicher - '93)

Let $c_1, \dots, c_n$ be exchangeable centered random variables. Suppose that $\tau(c_{i_1} \cdots c_{i_k}) = 0$ whenever there exists an indice i_l different from the rest. Then

$$S_n = \frac{1}{\sqrt{n}} \sum_{k \in [n]} c_k$$

converges in distribution and

$$\lim_{n \to \infty} \tau(S_n^p) = \sum_{\pi \in P_2(p)} \tau(\pi).$$

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$$\bigcup_{j \in [p/2]} V_j = [p],$$

   and $|V_j| = 2$ for each $j \in [p/2]$.
3. For a partition $\pi \in P(p)$, we define

$$\tau(\pi) = \tau(c_{i_1} \cdots c_{i_p}),$$

   where $i_k = i_j$ if and only if j, k are in the same block of π.

Centering of tensors

1. For $\delta c = \delta b = a \otimes a - \lambda^2$ and $\pi \in P(p)$, we have

$$\delta^p \tau \otimes \tau(\pi) = \sum_{I \subseteq [p]} (-1)^{|I|} \lambda^{2|I|} \tau^2 \left(\overrightarrow{\prod_{j \in I^c}} a_{i_j} \right).$$

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2. If $\{1\} \in \pi$, we have

$$\begin{aligned} \delta^p \tau \otimes \tau(\pi) &= \delta^p \tau \otimes \tau \left((a_{i_1} \otimes a_{i_1}) b_{i_2} \cdots b_{i_p} \right) - \lambda^2 \delta^p \tau \otimes \tau(b_{i_2} \cdots b_{i_p}) \\ &= 0. \end{aligned}$$

Decomposition of partitions

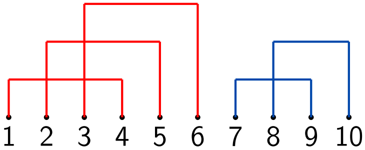
1. If π can be decomposed into small partitions, say, $\pi = \pi_1 \boxplus \pi_2$, then

$$\tau \otimes \tau(\pi) = \tau \otimes \tau(\pi_1)\tau \otimes \tau(\pi_2).$$

2. $\tau \otimes \tau(\pi)$ is then a multiplicative function on

$$P(\infty) := \bigcup_{n \in \mathbb{N}} P(n).$$

Decomposition of partitions



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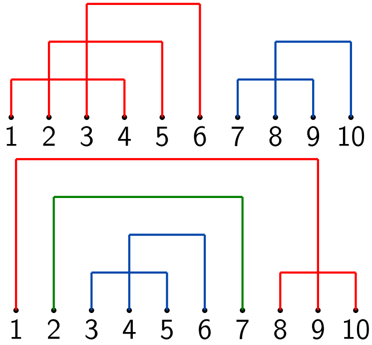

Type of partitions

1. We say that $\pi \in P_2(p)$ has a crossing if there exists $i < k < j < l$ such that $V = \{i, j\} \in \pi$ and $U = \{k, l\} \in \pi$. In this case, we say that V crosses U.

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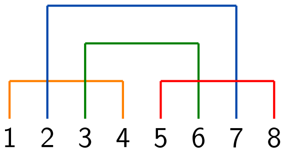
3. A partition is connected if it cannot be decomposed into nontrivial subpartitions. The set of connected pair partitions is denoted by $P_2^{\text{con}}(p)$.

Intersection graph

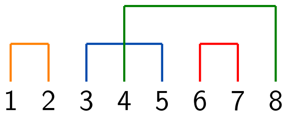
1. Given a partition $\pi = \{V_1, \ldots, V_{p/2}\} \in P_2(p)$, we define its intersection graph $G(\pi)$ as follows. The vertices of $G(\pi)$ are the blocks $V_1, \ldots, V_{p/2}$, and there exists an edge between V_i and V_j if they cross.
2. We denote $\mathrm{cc}(\pi)$ the number of connected components of $G(\pi)$, $\mathrm{ncr}(\pi)$ the number of isolated vertices of $G(\pi)$ and $\mathrm{cr}(\pi)$ the number of non-isolated vertices, that is,

$$\mathrm{cr}(\pi) + \mathrm{ncr}(\pi) = p/2.$$

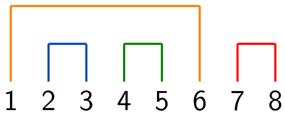
Examples



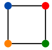
(a) Connected partition

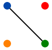


(b) General partition



(c) Noncrossing partition





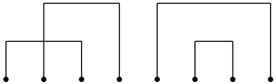
The free cumulants of pair partitions

Consider $\pi \in P_2(p)$. We decompose π into its connected components. Namely, let $\hat{\pi} \in NC(p)$ be the choice of connected components and for each $T \in \hat{\pi}$, we draw $\pi_T \in P_2^{\text{con}}(T)$. Hence

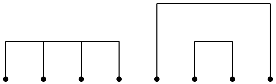
$$|P_2(p)| = \sum_{\hat{\pi} \in NC(p)} \prod_{T \in \hat{\pi}} |P_2^{\text{con}}(T)|.$$

The mapping $\Phi(\pi) = (\hat{\pi}, (\pi_T)_{T \in \hat{\pi}})$ is a bijection (Lehner - '01).

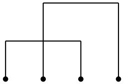
Bijection Φ



(a) π



(b) $\hat{\pi}$



(c) $\pi_{\{1,2,3,4\}}$

(d) $\pi_{\{5,8\}} \cong \pi_{\{6,7\}}$

Figure: A partition π and its image $\Phi(\pi)$.

Decomposition of $\tau(\pi)$

1. Using Φ and the fact that $\tau \otimes \tau(\pi)$ is multiplicative, we get

$$\sum_{\pi \in P_2(p)} \tau \otimes \tau(\pi) = \sum_{\hat{\pi} \in NC(p)} \prod_{T \in \hat{\pi}} \left(\sum_{\pi_T \in P_2^{\mathrm{con}}(T)} \tau \otimes \tau(\pi_T) \right).$$

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2. If $|T| = 2$, we have $\tau \otimes \tau(\pi_T) = 1$.

Corollary 1

If $\pi \in NC_2(p)$, we have $\tau \otimes \tau(\pi) = 1$.

Bipartite pair partitions

Definition 2

Let $\pi \in P_2(p)$. We say that π is a bipartite pair partition and is denoted by $\pi \in P_2^{\text{bi}}(p)$ if $G(\pi)$ is bipartite. We denote $\pi \in P_2^{\text{bicon}}(p)$ if π is connected and bipartite.

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Proposition 1

Let $\pi \in P_2^{\text{con}}(p)$, for even integer $p \geq 4$. Then, the following hold.

1. *If $\pi \notin P_2^{\text{bicon}}(p)$, then $\tau(\pi) = 0$.*
2. *If $\pi \in P_2^{\text{bicon}}(p)$, then*

$$\tau \otimes \tau(\pi) = 2 \left(\frac{q}{2}\right)^{p/2}.$$

Proof's outline

1. We label the blocks $V_1, \ldots, V_{p/2}$ of π such that V_j crosses at least one V_l, for $l < j$.

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1. We label the blocks $V_1, \ldots, V_{p/2}$ of π such that V_j crosses at least one V_l, for $l < j$.
2. We remove the blocks V_j one at a time.
3. How do we remove each block?
4. How does each block affect the rest of the blocks?

The first block

For simplicity, let us assume that $\delta = 1$, then

$$q = \frac{2\lambda^2}{\sigma^2 + 2\lambda^2} = \frac{2\sigma^2\lambda^2}{\sigma^2(\sigma^2 + 2\lambda^2)} = 2\sigma^2\lambda^2.$$

Let $V = \{1, v\} \in \pi$ be a block. Let $I_1 = \{2, \ldots, v-1\}$, $I_2 = \{v+1, \ldots, p\}$. Then

$$\tau \otimes \tau(\pi) = \tau \otimes \tau\left(b_{i_1} B_{I_1} b_{i_v} B_{I_2}\right).$$

Let $a_{i_v} = a$. Then

$$\tau \otimes \tau(\pi) = \tau \otimes \tau\left((a \otimes a) B_{I_1} (a \otimes a) B_{I_2}\right) - \lambda^4 \tau \otimes \tau\left(B_{I_1} B_{I_2}\right).$$

The first block

We then have

$$\tau \otimes \tau\left((a \otimes a) B_{I_1} (a \otimes a) B_{I_2}\right) = \sum_{\substack{J_1 \subseteq I_1 \\ J_2 \subseteq I_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau^2 \left(a a_{J_1^c} a a_{J_2^c}\right).$$

The variable a is free from $\{a_{J_1^c}, a_{J_2^c}\}$. Then

$$\tau\left(a a_{J_1^c} a a_{J_2^c}\right) = \sigma^2 \tau(a_{J_1^c}) \tau(a_{J_2^c}) + \lambda^2 \tau(a_{J_1^c} a_{J_2^c}).$$

The first block

- $\lambda^4 \sum_{\substack{J_1 \subseteq I_1 \\ J_2 \subseteq I_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau^2(a_{J_1^c} a_{J_2^c})$
- $\sigma^4 \sum_{\substack{J_1 \subseteq I_1 \\ J_2 \subseteq I_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau^2(a_{J_1^c}) \tau^2(a_{J_2^c})$
- $\sigma^2\lambda^2 \sum_{\substack{J_1 \subseteq I_1 \\ J_2 \subseteq I_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c}) \tau(a_{J_2^c}) \tau(a_{J_1^c} a_{J_2^c})$
- $\sigma^2\lambda^2 \sum_{\substack{J_1 \subseteq I_1 \\ J_2 \subseteq I_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c} a_{J_2^c}) \tau(a_{J_1^c}) \tau(a_{J_2^c})$

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- $\sigma^2 \lambda^2 \sum_{\substack{J_1 \subseteq I_1 \\ J_2 \subseteq I_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c}) \tau(a_{J_2^c}) \tau(a_{J_1^c} a_{J_2^c})$
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The first block

Let $(\tilde{a}_k)_{k\in\mathbb{N}}$ be another free i.i.d family of copies of a, free independent from $(a_k)_{k\in\mathbb{N}}$.
Let B_1 be a vector such that

$$(B_1)_j = a_{i_j} \otimes \tilde{a}_{i_j} - \lambda^2; \quad j \in I_1$$
$$(B_1)_j = a_{i_j} \otimes a_{i_j} - \lambda^2; \quad j \in I_2.$$

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Consider $\tau \otimes \tau(\pi \setminus V, V^{(1)})$ given by

$$\tau \otimes \tau(\pi \setminus V, V^{(1)}) = \tau \otimes \tau\left((B_1)_{I_1}(B_1)_{I_2}\right).$$

The first block

Let $(\tilde{a}_k)_{k\in\mathbb{N}}$ be another free i.i.d family of copies of a, free independent from $(a_k)_{k\in\mathbb{N}}$. Let B_1 be a vector such that

$$(B_1)_j = a_{i_j} \otimes \tilde{a}_{i_j} - \lambda^2; \quad j \in I_1$$
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For comparison,

$$\tau \otimes \tau(\pi \setminus V) = \tau \otimes \tau\left((B)_{I_1}(B)_{I_2}\right).$$

The first block

We deduce that

$$\tau \otimes \tau(\pi) = 2\sigma^2\lambda^2\tau \otimes \tau(\pi \setminus V, V^{(1)}) = q\tau \otimes \tau(\pi \setminus V, V^{(1)}).$$

In the first interaction, consider a block $U = \{u_1, u_2\}$ that crosses V, say $1 < u_1 < v$, then

1. Variables in V: $(a_{i_1} \otimes a_{i_1}, a_{i_v} \otimes a_{i_v}) - \lambda^2$;
2. Variables in U: $(a_{i_{u_1}} \otimes \tilde{a}_{i_{u_1}}, a_{i_{u_2}} \otimes a_{i_{u_2}}) - \lambda^2$.

Second block

Let $a = a_{i_{u_1}}$ and assume that $u_2 = 2$. Removing U now, we get

$$\tau \otimes \tau(\pi \setminus V, V^{(1)}) = \tau \otimes \tau\left((a \otimes \tilde{a})(B_1)_{I'_1}(a \otimes a)(B_1)_{I'_2}\right) - \lambda^4 \tau \otimes \tau\left((B_1)_{I'_1}(B_1)_{I'_2}\right).$$

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We obtain

$$\tau \otimes \tau \left((a \otimes \tilde{a})(B_1)_{I_1'} (a \otimes a)(B_1)_{I_2'} \right) = \sum_{\substack{J_1 \subseteq I_1' \\ J_2 \subseteq I_2'}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau \left(a a_{J_1^c} a a_{J_2^c} \right) \tau \left(\tilde{a} \bar{a}_{J_1^c} a a_{J_2^c} \right),$$

where $\bar{a}_{i_j} = \tilde{a}_{i_j}$ if $j \in I_1$ or $\bar{a}_{i_j} = a_{i_j}$ if I_2.

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where $\bar{a}_{i_j} = \tilde{a}_{i_j}$ if $j \in I_1$ or $\bar{a}_{i_j} = a_{i_j}$ if I_2. Recall that

$$\tau \left(a a_{J_1^c} a a_{J_2^c} \right) \tau \left(\tilde{a} \bar{a}_{J_1^c} a a_{J_2^c} \right) = \left(\sigma^2 \tau(a_{J_1^c}) \tau(a_{J_2^c}) + \lambda^2 \tau(a_{J_1^c} a_{J_2^c}) \right) \lambda^2 \tau(\bar{a}_{J_1^c} a_{J_2^c}).$$

Second block

- $\lambda^4 \sum_{\substack{J_1 \subseteq I'_1 \\ J_2 \subseteq I'_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c} a_{J_2^c}) \tau(\bar{a}_{J_1^c} a_{J_2^c})$
- $\lambda^2 \sigma^2 \sum_{\substack{J_1 \subseteq I'_1 \\ J_2 \subseteq I'_2}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c}) \tau(a_{J_2^c}) \tau(\bar{a}_{J_1^c} a_{J_2^c})$

Second block

- $\lambda^4 \sum_{\substack{J_1 \subseteq I_1' \\ J_2 \subseteq I_2'}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c} a_{J_2^c}) \tau(\bar{a}_{J_1^c} a_{J_2^c}) = \lambda^4 \tau \otimes \tau(\pi \setminus \{V, U\}, V^{(1)})$
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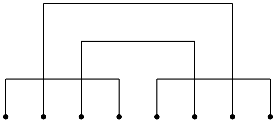
- $\lambda^4 \sum_{\substack{J_1 \subseteq I_1' \\ J_2 \subseteq I_2'}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c} a_{J_2^c}) \tau(\bar{a}_{J_1^c} a_{J_2^c}) = \lambda^4 \tau \otimes \tau(\pi \setminus \{V, U\}, V^{(1)})$
- $\lambda^2 \sigma^2 \sum_{\substack{J_1 \subseteq I_1' \\ J_2 \subseteq I_2'}} (-1)^{|J_1|+|J_2|} \lambda^{2(|J_1|+|J_2|)} \tau(a_{J_1^c}) \tau(a_{J_2^c}) \tau(\bar{a}_{J_1^c} a_{J_2^c}) = \frac{q}{2} \tau \otimes \tau(\pi \setminus \{V, U\}, V^{(1)}, U^{(0)})$.

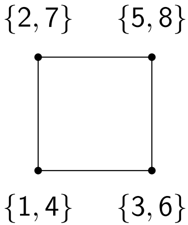
Hence

$$\tau \otimes \tau(\pi) = q \tau \otimes \tau(\pi \setminus V, V^{(1)}) = q \cdot \frac{q}{2} \tau \otimes \tau(\pi \setminus \{V, U\}, V^{(1)}, U^{(0)}).$$

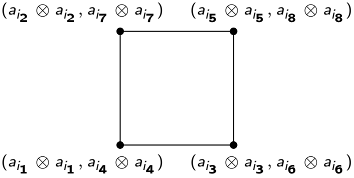
Illustration



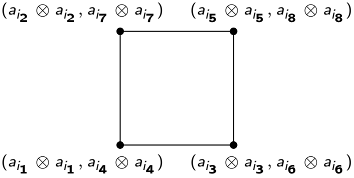
(a) Cycle C_4

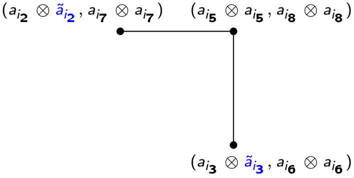


Illustration



Illustration





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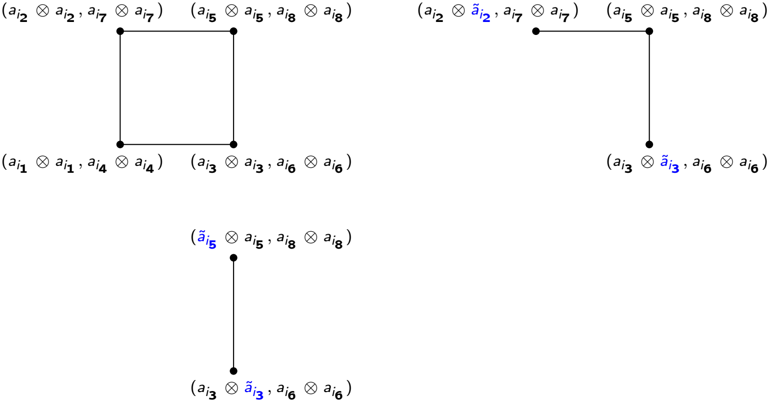
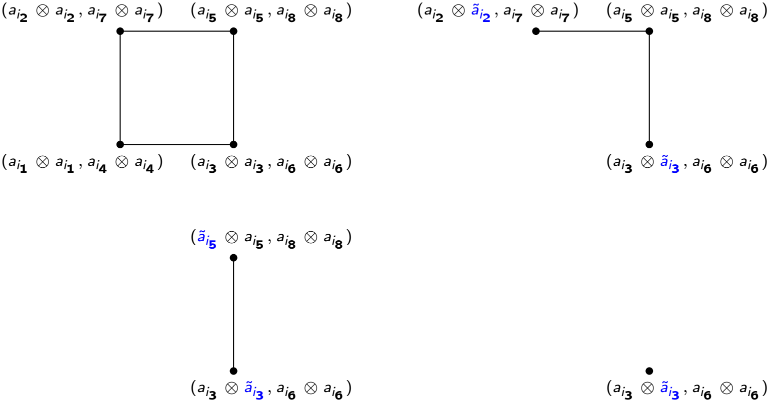
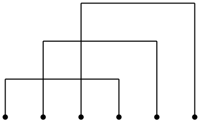


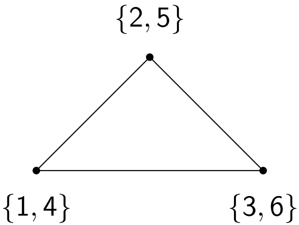
Illustration



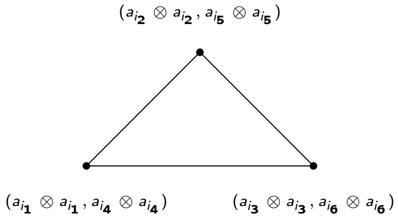
Illustration



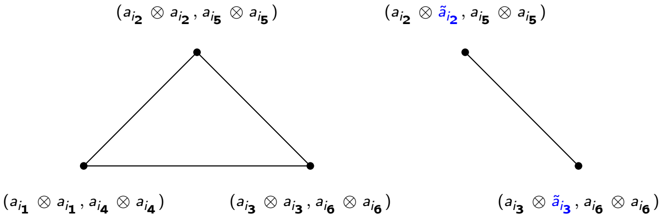
(a) Cycle C_3



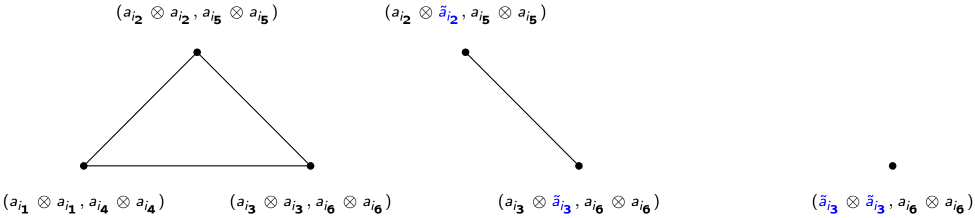
Illustration



Illustration



Illustration



Free cumulants

Let s_q be the limiting law. By Lehner's argument, its odd free cumulants $\kappa_n^{\text{free}}(s_q)$ vanish, $\kappa_2^{\text{free}}(s_q) = 1$ and for $n \geq 2$, we have

$$\kappa_{2n}^{\text{free}}(s_q) = 2\left(\frac{q}{2}\right)^n |P_2^{\text{bicon}}(2n)|.$$

Using the bijection Φ, it can be checked that

$$\tau(s_q^{2p}) = \sum_{\pi \in P_2^{\text{bi}}(2p)} 2^{\text{cc}(\pi)-p} q^{\text{cr}(\pi)}.$$

Limiting law

Recall that

$$\mu_q := \sqrt{q}\left(\frac{1}{\sqrt{2}}\mu_{sc} + \frac{1}{\sqrt{2}}\mu_{sc}\right) \boxplus \sqrt{1-q}\,\mu_{sc}.$$

Since the free cumulants are multilinear and linearize the free convolution, it suffices to check that s_1 has distribution μ_1.

Proposition 2

$$\tau\left((x_1 + x_2)^{2p}\right) = \sum_{\pi \in P_2^{\mathrm{bi}}(2p)} 2^{\mathrm{cc}(\pi)},$$

where x_1, x_2 are classical i.i.d semi-circle random variables.

Combinatorics of bipartite pair partitions

1. Firstly, we have

$$\tau\left((x_1+x_2)^{2p}\right)=\sum_{l=0}^{p}\binom{2p}{2l}C_lC_{p-l}=C_pC_{p+1}.$$

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2. Note that

$$\sum_{I \subseteq [2p]} |\{(\pi_1, \pi_2) : \pi_1 \in NC_2(I), \pi_2 \in NC_2(I^c)\}| = \sum_{l=0}^{p} \binom{2p}{2l} C_l C_{p-l}.$$

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3. We must prove that

$$\sum_{\pi \in P_2^{\text{bi}}(2p)} 2^{\text{cc}(\pi)} = \sum_{I \subseteq [2p]} |\{(\pi_1, \pi_2) : \pi_1 \in NC_2(I), \pi_2 \in NC_2(I^c)\}|$$

Combinatorics of bipartite pair partitions

1. Let $\pi \in P_2^{\mathrm{bi}}(2p)$. We can find bipartite sets $\pi_1 = \{V_1, \ldots, V_m\}, \pi_2 = \{V_{m+1}, \ldots, V_p\}$ of π such that $V_1, \ldots, V_m$ do not cross, neither do $V_{m+1}, \ldots, V_p$.

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$$\pi_1 \in NC_2(I);$$
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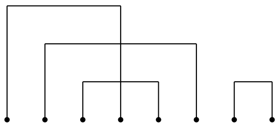
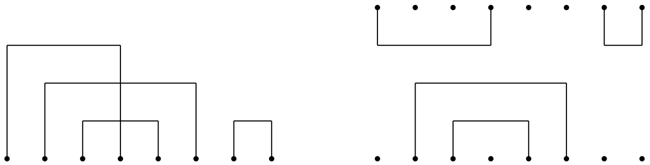
$$I = \bigcup_{i=1}^{m} V_i.$$

3. We call (π_1, π_2, I, I^c) a noncrossing representation of π.

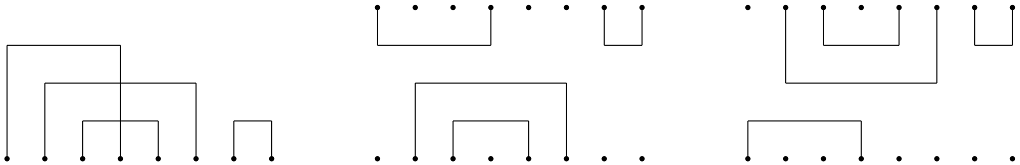
Combinatorics of bipartite pair partitions



Combinatorics of bipartite pair partitions



Combinatorics of bipartite pair partitions



Combinatorics of bipartite pair partitions

The number of noncrossing representations of π is equal to $2^{\mathrm{cc}(\pi)}$. Hence

$$\sum_{\pi \in P_2^{\mathrm{bi}}(2p)} 2^{\mathrm{cc}(\pi)} = |\{(\pi_1, \pi_2, I, I^c) : I \subseteq [2p], \pi_1 \in NC_2(I), \pi_2 \in NC_2(I^c)\}|.$$

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Remark 6.1

$$\kappa_{2n}^{\mathrm{free}}(x_1 + x_2) = 2|P_2^{\mathrm{bicon}}(2n)|.$$

Free probability

1. Higher-order tensors: A CLT for

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Free probability

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$$S_n^L := \frac{1}{\sqrt{n}} \sum_{k \in [n]} a_k^{\otimes L} - \lambda^L.$$

2. ϵ-independent settings [Mlotkowski - '04]. Consider $(a^{(i)})_{k\in\mathbb{N}} \in \mathcal{A}_i$ free i.i.d random variables. If $\mathcal{A}_1, \ldots, \mathcal{A}_L$ are ϵ-independent subalgebras, what is the central limit theorem for the variables

$$b_k := \overrightarrow{\prod_{l \in [L]}} a_k^{(l)}?$$

Combinatorics

3. We computed that

$$\kappa_{2n}^{\text{free}}(x_1 + x_2) = 2|P_2^{\text{bicon}}(2n)|.$$

Combinatorics

4. We computed that

$$\kappa_{2n}^{\text{free}}(x_1 + x_2) = 2|P_2^{\text{bicon}}(2n)|.$$

This suggests that there exists a measure ν such that

$$\int x^{2p}\,\mathrm{d}\nu = |P_2^{\text{bi}}(2p)|.$$

In this case, we can write $\mu_{sc} + \mu_{sc} = \nu \boxplus \nu$. This is the moment problem for $a_n = |P_2^{\text{bi}}(2n)|$.

Combinatorics

5. A q-Gaussian measure μ_q [Bozejko, Speicher, Kümmerer, Buchholz] is defined via

$$\int x^{2p}\,\mathrm{d}\mu_q = \sum_{\pi \in P_2(2p)} q^{\mathrm{crossings}(\pi)}.$$

When $q = 0$, we have $\mu_0 = \mu_{sc}$, and when $q = 1$, we have $\mu_1 = N(0,1)$.

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When $q = 0$, we have $\mu_0 = \mu_{sc}$, and when $q = 1$, we have $\mu_1 = N(0,1)$.We could consider an integer interpolation instead, which is defined as follows. Consider $P_2^{(m)}(2p)$ the set of m-colorable pair partitions and define

$$\int x^{2p}\,\mathrm{d}\mu_m := |P_2^{(m)}(2p)|.$$

We have $\mu_1 = \mu_{sc}$ and $\mu_\infty = N(0,1)$. Also, $\mu_2 = \nu$.

Thank you!