The Boolean quadratic forms and tangent law

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Plan of the talk

1. Free probability
2. Free central limit theorems for quadratic forms
3. Boolean probability
4. Boolean central limit theorems for quadratic forms

Basic notation

We consider a non commutative probability space $(\mathcal{A}, \tau)$. $\mathcal{A}$ is an algebra τ is a linear functional such that

1. τ is linear, weak*-continuous,
2. $\tau(\mathbb{I}) = 1$ - normal,
3. $\tau(XX^*) \geq 0$ - positive,
4. $\tau(XX^*) = 0$ implies $X = 0$ - faithful.

Distribution

A (noncommutative) random variable X is a self-adjoint element of $\mathcal{A}$ with a probability measure μ on $\mathbb{R}$ such as

$$\tau(X^n) = \int_{\mathbb{R}} x^n \mu(dx).$$

Free independence

We say that subalgebras $\mathcal{A}_i \subset \mathcal{A}$ are free independent if for every choice of $i_1 \neq i_2 \ldots \neq i_n$ and every choice of $X_i \in \mathcal{A}_i$ such that $\tau(X_i) = 0$ we have

$$\tau(X_1 X_2 \ldots X_n) = 0,$$

Example

Let G be the free product of groups $G_i, i \in I$, let $A = \mathbb{C}G$ be the group algebra,

$\phi(g) = \delta_{g=e}$ be the group trace, and set $A_i = \mathbb{C}G_i \subset A$.

Then $A_i : i \in I$ are free independent.

Let μ and ν be probability measures on $\mathbb{R}$, and X, Y self-adjoint free random variables with respective distributions μ and ν.

The distribution of $X + Y$ is called the free additive convolution of μ and ν and is denoted by $\mu \boxplus \nu$.

The analytic approach to free convolution is based on the Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-y}\mu(dy).$$

For measures with compact support the Cauchy transform is analytic at infinity and related to the moment generating function M_X as follows:

$$M_X(z) = \sum_{n=0}^{\infty} \tau(X^n)\, z^n = \frac{1}{z} G_X(1/z).$$

The Cauchy transform has an inverse in some neighbourhood of infinity which has the form

$$G_{\mu}^{-1}(z) = \frac{1}{z} + R_{\mu}(z),$$

where $R_{\mu}(z)$ is analytic in a neighbourhood of zero and is called *R-transform*.

The coefficients of its series expansion

$$R_X(z) = \sum_{n=0}^{\infty} K_{n+1}(X)\, z^n.$$

are called *free cumulants* of the random variable X.

The free convolution can now be computed via the identity

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$

The Wigner semicircle law has density

$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, dx$$

on $-2 \leq x \leq 2$. Its Cauchy-Stieltjes transform is given by the formula

$$G_\mu(z) = \frac{z - \sqrt{z^2 - 4}}{2}$$

and the R-transform is

$$R_X(z) = z.$$

We say that a sequence X_n of random variables *converges in distribution* towards X as $n \to \infty$, denoted by

$$X_n \xrightarrow{distr} X$$

if we have for all $m \in \mathbb{N}$

$$\lim_{n\to\infty} \tau(X_n^m) = \tau(X^m).$$

Let X_i be free copies of a centered random variable X of variance 1, then the sequence of quadratic forms

$$Q_n = \frac{1}{n}\sum_{i,j=1}^{n} a_{i,j}^{(n)} X_i X_j$$

converges in distribution to a compound free Poisson distribution with jump distribution μ.

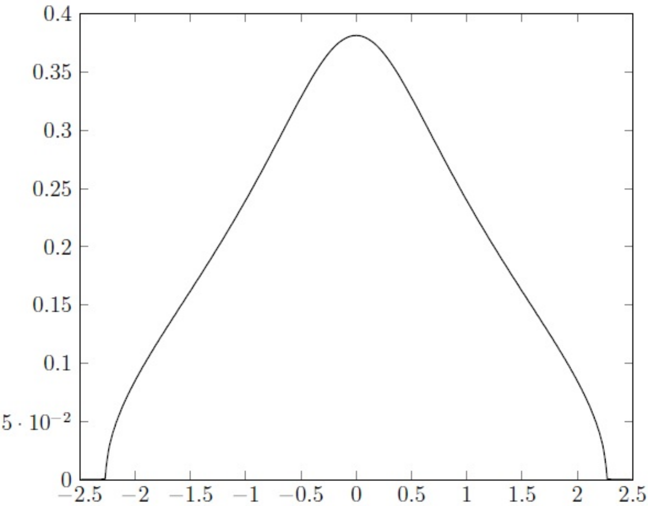
The free tangent law, W. Ejsmont and F. Lehner 2000

Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be free centred identically distributed random variables of variance 1, then

$$Q_n = \frac{1}{n} \sum_{k,j=1}^{n} i(X_k X_j - X_j X_k) \xrightarrow{distr} Y,$$

where $R_Y(z) = \tan(z)$.

It is worth mentioning that one of the fundamental examples of Nevanlinna functions is the tangent function, see Donoghue, 1974.



Density of the free tangent law

Proof 1.

Let

$$Q_n = \frac{1}{n} \sum_{i,j=1}^{n} a_{i,j}^{(n)} X_i X_j$$

where X_i are free centred identically distributed random variables of variance 1.
Let

$$Q'_n = \frac{1}{n} \sum_{i,j=1}^{n} a_{i,j}^{(n)} X_i X_j$$

where X_i sequence of standard free semicircular variables.
Then

$$lim_{n\to\infty} K_r(Q_n) = lim_{n\to\infty} K_r(Q'_n)$$

Proof 1.

We may assume without loss of generality that X_i are standard free semicircular variables.

The cumulants can be computed using formula of Leonov/Shiryaev and evaluate to

$$K_r\Big(\sum_{\substack{k,j=1\\k<j}}^{n} i(X_kX_j - X_jX_k)\Big) = \mathrm{Tr}(A_n^r) \text{ where } A_n = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}_n,$$

hence the odd cumulants vanish.

The eigenvalues of the matrix A_n are $\lambda_k = \cot\left(\frac{\pi}{2n} + \frac{k}{n}\pi\right)$ for $k \in \{0, \dots, n-1\}$ hence the even cumulants evaluate to

$$K_{2m}\Big(\sum_{\substack{k,j=1\\k<j}}^{n} i(X_kX_j - X_jX_k)\Big) = \sum_{k=0}^{n-1} \cot^{2m}\left(\frac{\pi}{2n} + \frac{k}{n}\pi\right)$$

$$= n^{2m}\frac{T_{2m-1}}{(2m-1)!} + \mathcal{O}(n^{2m-2})$$

where T_{2m-1} are the tangent numbers

$$\tan z = \sum_{n=1}^{\infty} T_n \frac{z^n}{n!} = z + \frac{2}{3!}z^3 + \frac{16}{5!}z^5 + \frac{272}{7!}z^7 + \cdots,$$

Hence

$$\lim_{n\to\infty} K_{2m}(Q_n) = \frac{T_{2m-1}}{(2m-1)!}$$

and we conclude that

$$\lim_{n\to\infty} R_{Q_n}(z) = \tan(z).$$

Proof 2.

The matrix $\frac{1}{n}A_n = \frac{1}{n}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}_n$ has characteristic polynomial

$$\chi_n(\lambda) = \frac{i(\lambda - \frac{i}{n})^n + i(\lambda + \frac{i}{n})^n}{2i}.$$

The cumulant generating function

$$R_{Q_n}(z) = \sum_{k=1}^{\infty} \frac{\mathrm{Tr}(A_n^k)}{n^k} z^{k-1},$$

can be obtained from the logarithmic derivative of the characteristic polynomial.

Indeed if we factorize the characteristic polynomial $\chi_n(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$ then

$$\frac{\chi_n'(\lambda)}{\chi_n(\lambda)} = \sum_{i=1}^{n} \frac{1}{\lambda - \lambda_i}$$

and

$$\frac{1}{z}\frac{\chi_n'(1/z)}{\chi_n(1/z)} = \sum_{k=0}^{\infty}\sum_{i=1}^{n} \lambda_i^k z^k = n + zR_{Q_n}(z).$$

In our case

$$\lim_{n\to\infty} R_{Q_n}(z) = \lim_{n\to\infty} \frac{1}{z}\left(\frac{1}{z}\frac{\chi_n'(1/z)}{\chi_n(1/z)} - n\right)$$
$$= \tan(z).$$

The free zigzag law

Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ be free centred identically distributed random variables of variance 1,, then

$$Q_n = \frac{1}{2n} \sum_{\substack{k,l=1 \\ k<l}}^{n} \left(X_k X_l + X_l X_k + i(X_k X_l - X_l X_k) \right) \xrightarrow{distr} Y,$$

where $R_Y(z) = \frac{1}{2}(\tan(z) + \sec(z) - 1)$.

An alternating permutation (or zigzag permutation) of the set $\{1, 2, 3, ..., n\}$ is an arrangement of those numbers so that each entry is alternately greater or less than the preceding entry.

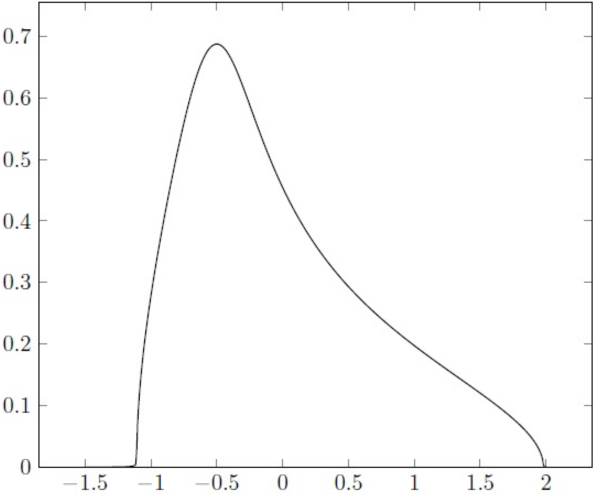
For example, the five alternating permutations of 1, 2, 3, 4 are:

- 1, 3, 2, 4 because $1 < 3 > 2 < 4$,
- 1, 4, 2, 3 because $1 < 4 > 2 < 3$,
- 2, 3, 1, 4 because $2 < 3 > 1 < 4$,
- 2, 4, 1, 3 because $2 < 4 > 1 < 3$,
- 3, 4, 1, 2 because $3 < 4 > 1 < 2$.

The determination of the number E_n of alternating permutations of the set $\{1, ..., n\}$ is called André's problem.

Theorem (Désiré André, 1879)

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} = \tan(z) + \sec(z).$$



Density of the free zigzag law

Free generalized tangent law

Let $X_1, X_2, \dots, X_n \in \mathcal{A}_{sa}$ be free centred identically distributed random variables of variance 1,, then for any $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $b \neq 0$, the limit law

$$Q_n = \frac{1}{n} \sum_{\substack{k,j=1 \\ k<j}}^{n} \left(a(X_k X_j + X_j X_k) + ib(X_k X_j - X_j X_k)\right) \xrightarrow{distr} Y,$$

has R-transform

$$R_Y(z) = \frac{\tan(bz)}{b - a\tan(bz)}.$$

A tangent number $T_n^{(k+1)}$ of order $k+1$ defined by

$$\tan^{k+1}(z) = \sum_{n=k+1}^{\infty} \frac{T_n^{(k+1)} z^n}{n!}$$

Carlitz and Scoville's [Duke Math. J. 39 (1972)]

$$\frac{\tan(z)}{1 - x\tan(z)} = \sum_{n=1}^{\infty} P_n(x)\frac{z^n}{n!}$$

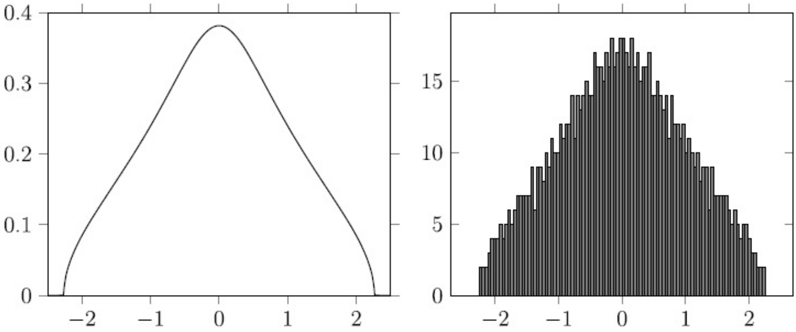
where $P_n(x) = \sum_{k=0}^{n-1} T_n^{(k+1)} x^k$

Free generalized tangent law

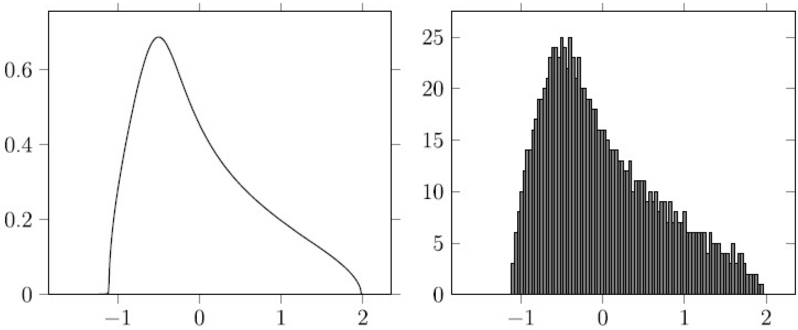
$$R_Y(z) = \frac{\tan(bz)}{b - a\tan(bz)} = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} T_n^{(k+1)} a^k b^{n-1-k} \right) \frac{z^n}{n!}$$

$$K_n = \frac{\sum_{k=0}^{n-1} T_n^{(k+1)} a^k b^{n-1-k}}{n!}$$

Random matrix model



Random matrix model



Random matrix model

Let $X_{N\times NM}$ be a complex Gaussian random matrix of size $N \times NM$
and let

$$A_M = \begin{bmatrix} 0 & a+ib & a+ib & \dots & a+ib \\ a-ib & 0 & a+ib & \dots & a+ib \\ a-ib & a-ib & 0 & \ddots & a+ib \\ \vdots & & \ddots & \ddots & \\ a-ib & a-ib & a-ib & \dots & 0 \end{bmatrix}.$$

Random matrix model

Let P_N be a sequence of $N \times N$ deterministic matrices all of whose moments with respect to the normalized trace converge to 1, e.g., the identity matrices $P_N = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]_N$ or any projection matrix of large rank like $P_N = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]_N - \frac{1}{N}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]_N$, then the spectral measures of

$$\frac{1}{M} X_{N \times NM}[A_M \otimes P_N] X^*_{N \times NM}$$

converge in distribution to the free generalized tangent law

$$R(z) = \frac{\tan(bz)}{b - a\tan(bz)}.$$

Boolean Independence.

We say that subalgebras $\mathcal{A}_i \subset \mathcal{A}$ are Boolean independent if for every choice of $i_1 \neq i_2 \ldots \neq i_n$ and every choice of $X_i \in \mathcal{A}_i$, we have

$$\tau(X_1 X_2 \ldots X_n) = \tau(X_1)\tau(X_2)\ldots\tau(X_n)$$

Example

The most natural state on the group algebra of the free group $\mathrm{F_N}$ with the free generators x_1 , $x_2, \ldots,$ x_N is the Haagerup state

$$H_q(g) = q^{l(g)}, \quad g \in \mathrm{F_N}, q \in (0,1)$$

where $g = x_{i_1}^{n_1} x_{i_2}^{n_2} \ldots x_{i_k}^{n_k}$, $l(g) = \sum_i |n_i|$ and $l(e) = 0$.

Then group subalgebras $C[G_i]$ are Boolean independent where G_i are cyclic group generated by x_i.

Let μ and ν be probability measures on $\mathbb{R}$, and X, Y self-adjoint Boolean independent random variables with respective distributions μ and ν.

The distribution of $X + Y$ is called the Boolean additive convolution of μ and ν and is denoted by $\mu \uplus \nu$.

The analytic approach to Boolean convolution is based on the Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-y}\mu(dy).$$

For measures with compact support the Cauchy transform is analytic at infinity and related to the moment generating function M_X as follows:

$$M_X(z) = \sum_{n=0}^{\infty} \tau(X^n)\, z^n = \frac{1}{z} G_X(1/z).$$

Moreover, the moment generating transform can be written as

$$M_{\mu}(z) = \frac{1}{1 - H_{\mu}(z)},$$

where $H_{\mu}(z)$ is analytic in a neighbourhood of zero. The coefficients of its series expansion

$$H_X(z) = \sum_{n=1}^{\infty} K_n(X) z^n$$

are called *Boolean cumulants* of the random variable X.

Boolean Gaussian distribution

The Boolean Gaussian law with mean zero and variance a^2 has distribution

$$\frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_{a}.$$

Boolean Gaussian distribution

In contrast to the classical convolution, it is not true that, for arbitrary $a \in \mathbb{R}$, the convolution $\mu \uplus \delta_c$ is equal to the shift of measure μ by the amount c.

For example, $(\frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a) \uplus \delta_c$ is equal to

$$\frac{\left(1 + \frac{c}{\sqrt{4a^2+c^2}}\right)\delta_{(c+\sqrt{4a^2+c^2})/2} + \left(1 - \frac{c}{\sqrt{4a^2+c^2}}\right)\delta_{(c-\sqrt{4a^2+c^2})/2}}{2}.$$

Boolean Gaussian distribution

We say that a family X_i $i \in \{1, \ldots, n\}\}$ is a *Boolean standard normal family* if $K_1(X_i) = K_2(X_i) = 1$, $K_r(X_i) = 0$ for $r > 2$ and X_i are Boolean independent

Boolean generalized tangent law, Ejsmont Hęćka

Let $X_1, \ldots, X_n$ be identically distributed Boolean independent random variables with mean $\tau(X_i) = \frac{1}{\sqrt{n}}$ and variance 1. Then the sequence of quadratic forms

$$Q_n = \frac{1}{n} \sum_{\substack{k,j=1 \\ k<j}}^{n} \left(a(X_k X_j + X_j X_k) + ib(X_k X_j - X_j X_k)\right) \xrightarrow{d} Y, \quad a, b \in \mathbb{R}$$

where the H-transform of the limit distribution has the form

$$H_Y(z) = \frac{1}{z} \frac{\tan(bz)}{b - a\tan(bz)} - 1.$$

The limit Boolean and free cumulants are not exactly the same

$$K_r^{\text{Boolean}} = K_{r+1}^{\text{free}}$$

The main problem

Let $X_1, \ldots, X_n$ be identically distributed Boolean independent normal random variables and

$$Q_n = \frac{1}{n} \sum_{\substack{k,j=1 \\ k<j}}^{n} \left(a(X_k X_j + X_j X_k) + ib(X_k X_j - X_j X_k) \right)$$

In the case of the Boolean normal distribution, with mean c, variance 1 takes the particular form

$$K_r(Q_n) = c^2 \operatorname{Tr}(JA_n^r)$$

where

$$J := \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \ddots & 1 \\ \vdots & & \ddots & \ddots & \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

$$A_n := \begin{bmatrix} 0 & a+ib & a+ib & \dots & a+ib \\ a-ib & 0 & a+ib & \dots & a+ib \\ a-ib & a-ib & 0 & \ddots & a+ib \\ \vdots & & \ddots & \ddots & \\ a-ib & a-ib & a-ib & \dots & 0 \end{bmatrix}.$$

Now we focus on the special case $a = 0$ and $b = 1$ because in this situation we are able to determine the corresponding measures.

We call the limit law μ_Y the Boolean tangent law if

$$H_Y(z) = \frac{\tan z}{z} - 1.$$

In this case the corresponding transforms are given by

$$M_\mu(z) = \frac{z}{2z - \tan(z)}, \quad G_\mu(z) = \frac{1}{2z - z^2 \tan(1/z)}.$$

If we use the Stieltjes inversion formula, namely

$$d\mu(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im} G_\mu(x+i\epsilon) = 0 \quad \text{for } 2/x \neq \tan(1/x) \text{ and } x \neq 0.$$

Thus the measure μ has no absolutely continuous part.

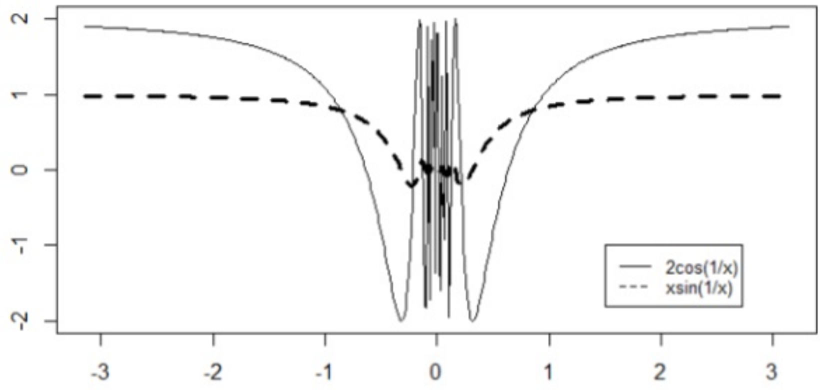
In order to determine the atoms, we compute the limits

$$\lim_{\epsilon \to 0^+} i\epsilon G_\mu (x + i\epsilon) \quad \text{for } \ 2/x = \mathtt{tan}(1/x) \text{ and } x = 0.$$

Finally, we get

$$\mu(\{x\}) = \begin{cases} \frac{x^2}{4-x^2} & \text{for } x \in \{x \mid 2/x = \tan(1/x)\}, \\ \frac{1}{2} & \text{for } x = 0. \end{cases}$$

$$2/x = \tan(1/x) \iff 2\cos(1/x) = x\sin(1/x)$$



Thank you for your attention