On Voiculescu's Topological Free Entropy for C*-algebras

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Thank you, Prof. Voiculescu, for inspiring my research work.

Outline

Voiculescu's Free Entropy Theory for Finite von Neumann Algebras

In the early 1980s, D. Voiculescu began the development of the theory of free probability and free entropy. This new and powerful tool was crucial in solving some old open problems in the field of von Neumann algebras.

A noncommutative W$^*-$*probability space* is a pair $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a finite von Neumann algebra and τ is a faithful normal tracial state. Elements of $\mathcal{M}$ are called *random variables*.

An example of finite von Neumann algebra

Let $M_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$ and τ_k be the normalized trace on $M_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k}\text{Tr}$, where Tr is the usual trace on $M_k(\mathbb{C})$. Let $M_k^{sa}(\mathbb{C})$ denote the self-adjoint complex matrices.

Let

$$(M_k^{sa}(\mathbb{C}))^n = M_k^{sa}(\mathbb{C}) \times M_k^{sa}(\mathbb{C}) \times \ldots M_k^{sa}(\mathbb{C}).$$

The euclidean norm $\| \ \|_e$ and 2-norm on $(M_k^{sa}(\mathbb{C}))^n$:

$$\|(A_1, \ldots, A_n)\|_e^2 = \text{Tr}\,(A_1^2 + \cdots + A_n^2),$$

and

$$\|(A_1, \ldots, A_n)\|_2^2 = \tau_k(A_1^2 + \cdots + A_n^2)$$

for each $(A_1, \ldots, A_n)$ in $(M_k^{sa})^n$. Let Λ denote the Lebesgue measure on $(M_k^{sa})^n$ induced by the euclidean norm $\| \ \|_e$.

The set of $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$

Let $(\mathcal{M}, \tau)$ be a W^*-probability space,

$$x_1, \ldots, x_n$$

be self-adjoint elements in $\mathcal{M}$. For $\epsilon, R > 0$, $m, k \in \mathbb{N}$, let

$$\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$$

be a subset of $(M_k^{sa})^n$ consisting of all

$$(A_1, \ldots, A_n)$$

in $(M_k^{sa})^n$ such that

$$|\tau(x_{i_1} \ldots x_{i_p}) - \tau_k(A_{i_1} \ldots A_{i_p})| < \epsilon,$$

for all $1 \le p \le m$, $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$ and $\|A_j\| \le R, 1 \le j \le m$.

Question 1.1

For any fixed R, m, ϵ, when k is large enough, is

$$\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$$

non-empty?

The answer to this question is equivalent to the answer to whether

$$W^*(x_1, \ldots, x_n) \hookrightarrow \mathcal{R}^\omega,$$

where $\mathcal{R}$ is the hyperfinite II_1 factor?

Free entropy

Then, we define successively,

$$\chi_R(x_1,\ldots,x_n;m,k,\epsilon) = \log\Lambda(\Gamma_R(x_1,\ldots,x_n;m,k,\epsilon)),$$

$$\chi_R(x_1,\ldots,x_n;m,\epsilon) = \limsup_{k\to\infty}\left(k^{-2}\chi_R(x_1,\ldots,x_n;m,k,\epsilon)+\frac{n}{2}\log k\right),$$

$$\chi_R(x_1,\ldots,x_n) = \inf\left\{\chi_R(x_1,\ldots,x_n;m,\epsilon) : m\in\mathbb{N},\epsilon>0\right\},$$

$$\chi(x_1,\ldots,x_n) = \sup_{R>0}\chi_R(x_1,\ldots,x_n).$$

Some basic properties of free entropy

Voiculescu proved the following basic properties of $\chi(x_1, \ldots, x_n)$,

- Upper Bound
$$\chi(x_1, \ldots, x_n) \leq 2^{-1} n \log(2\pi e n^{-1} C^2)$$
where $C^2 = \tau(x_1^2 + \cdots + x_n^2)$. In particular $\chi(x_1, \ldots, x_n)$ is either finite or $-\infty$.
- One Variable Case
$$\chi(x) = \iint \log|s - t| d\mu(s) d\mu(t) + \frac{3}{4} + \frac{1}{2} \log 2\pi,$$
where μ is the distribution of x.
- Assume $\chi(x_j) > -\infty, 1 \leq j \leq n$. Then,
$$\chi(x_1, \ldots, x_n) = \chi(x_1) + \cdots + \chi(x_n)$$
iff $x_1, \ldots, x_n$ are freely independent.

Free entropy dimension

Applications of free entropy can be nicely stated in the language of free entropy dimension (also developed by the D. Voiculescu).

Definition 2.1

Let $x_1, \ldots, x_n, y_1, \ldots, y_m$ be selfadjoint random variables in $(\mathcal{M}, \tau)$. The modified free entropy dimension *is defined by*

$$\delta_0(x_1, \ldots, x_n : y_1, \ldots, y_m) = n + \limsup_{\epsilon \to 0} \frac{\chi(x_1 + \epsilon S_1, ..., x_n + \epsilon S_n : S_1, ..., S_n, y_1, ..., y_m)}{|\log \epsilon|}$$

where $\{S_1, \ldots, S_n\}$ is a semicircular family and $x_1, \ldots, x_n, y_1, \ldots, y_m$ and $\{S_1, \ldots, S_n\}$ are free. If $m = 0$, we write $\delta_0(x_1, \ldots, x_n)$.

Free Entropy Dimension of a finite von Neumann Algebra

Definition 2.2

Free entropy dimension of a finite von Neumann algebra M, fdim(M) is defined as

$$fdim(M) = \sup\{\delta_0(x_1,\ldots,x_n), \\ x_1 \ldots x_n \text{ is a family of generators of } M\}$$

Question 2.3

- *Is every separable von Neumann algebra finitely generated? Singly generated?*
- *Does $\delta_0(x_1,\ldots,x_n) = \delta_0(y_1,\ldots,y_m)$, where $x_1,\ldots,x_n$ and $y_1,\ldots,y_m$ are two families of generators for a von Neumann algebra $\mathcal{M}$?*

- Voiculescu showed that, for any n in $\mathbb{N}$

$$\text{fdim}(L(F(n))) \geq n.$$

- In 1996, Voiculesu shows that if a finite von Neumann algebra M has Cartan subalgebras, then

$$\text{fdim}(M) \leq 1.$$

- In 1999, Ge showed that if a finite von Neumann algebra M is not prime, i.e., is a tensor product of two infinite-dimensional von Neumann algebras, then

$$\text{fdim}(M) \leq 1.$$

An equivalent definition of free entropy dimension by K. Jung

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$.For every $\omega > 0$, we define the ω-$\|\cdot\|_2$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|_2)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega.$$

Definition 2.4

Suppose that Σ is a set in $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_2(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|_2$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

Then, we define successively, $\forall \omega > 0$,

$$\delta_0(x_1, \ldots, x_n; R, \omega) = \inf_{m,\epsilon} \limsup_{k\to\infty} \frac{\log \nu_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon), \omega)}{-k^2 \log \omega}$$

$$\delta_0(x_1, \ldots, x_n) = \limsup_{\omega\to 0^+} \sup_{R>0} \delta_0(x_1, \ldots, x_n; R, \omega).$$

$\delta_0(x_1, \ldots, x_n)$ is called Voiculescu's free entropy dimension (here we use an equivalent definition by K. Jung).

Some other Results

- Absence of simple masa in free group factor by Ge; Absence of an abelian subalgebra with finite multiplicity in in free group factor by Dykema.
- Voiculescu (1999) showed that $\delta_0(x_1, \ldots, x_n) \leq 1$, where $x_1, \ldots, x_n$ is a family of natural generators for $L(SL_{2n+1}(\mathbb{Z}))$
- Ge and Shen (2002) showed that $\delta_0(x_1, \ldots, x_n) \leq 1$, where $x_1, \ldots, x_n$ is any family of generators for $L(SL_{2n+1}(\mathbb{Z}))$
- Jung and Shlyakhtenko (2007) showed that $\delta_0(x_1, \ldots, x_n) \leq 1$, where $x_1, \ldots, x_n$ is any family of generators for Proprty T factors.
- Many more by Jung, Shlyakhtenko and etc....

Free orbit diemnsion for finite von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra with a tracial state τ, and $x_1, \ldots, x_n$ be elements in $\mathcal{M}$. For any positive R and ϵ, and any m, k in $\mathbb{N}$, recall that $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$ is the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that $\|A_j\| \le R$, $1 \le j \le n$, and

$$|\tau_k(A_{i_1}^{\eta_1} \cdots A_{i_q}^{\eta_q}) - \tau(x_{i_1}^{\eta_1} \cdots x_{i_q}^{\eta_q})| < \epsilon,$$

for all $1 \le i_1, \ldots, i_q \le n$, all $\eta_1, \ldots, \eta_q$ in $\{1, *\}$, and all q with $1 \le q \le m$.

Observation 3.1

Assume $(A_1, \ldots, A_n) \in \Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$. For unitary matrix W,

$$(WA_1W^*, \ldots, WA_nW^*) \in \Gamma_R(x_1, \ldots, x_n; m, k, \epsilon).$$

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$. For every $\omega > 0$, we define the ω-orbit-ball $\mathcal{U}(B_1, \ldots, B_n; \omega)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \ldots, A_n) - (WB_1W^*, \ldots, WB_nW^*)\|_2 < \omega.$$

For $\omega > 0$, we define the ω-orbit covering number $o_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon), \omega)$ to be the minimal number of ω-orbit-balls that cover $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$ with the centers of these ω-orbit-balls in $\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon)$. Now we define, successively,

$$\mathfrak{K}_2(x_1, , \ldots, x_n; \omega, R) = \inf_{m \in \mathbb{N}, \epsilon > 0} \limsup_{k \to \infty} \frac{\log(o_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon), \omega))}{-k^2 \log \omega}$$

$$\mathfrak{K}_2(x_1, , \ldots, x_n; \omega) = \sup_{R > 0} \mathfrak{K}_2(x_1, , \ldots, x_n; \omega, R)$$

$$\mathfrak{K}_2(x_1, , \ldots, x_n) = \sup_{0 < \omega < 1} \mathfrak{K}_2(x_1, , \ldots, x_n; \omega),$$

where $\mathfrak{K}_2(x_1, , \ldots, x_n)$ is called the *free orbit-dimension* of $x_1, \ldots, x_n$.

Definition 3.2 (Hadwin & Shen (2007))

Suppose $\mathcal{M}$ is a finitely generated von Neumann algebra with a tracial state τ. Then the free orbit-dimension $\mathfrak{K}_2(\mathcal{M})$ of $\mathcal{M}$ is defined by

$$\mathfrak{K}_2(\mathcal{M}) = \sup\{\mathfrak{K}_2(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \text{ generate } \mathcal{M} \text{ as a von Neumann algebra}\}.$$

Key Properties of $\mathfrak{K}_2$

Lemma 3.3 (Hadwin & Shen (2007))

Let $x_1, \ldots, x_n$ be self-adjoint elements in a von Neumann algebra $\mathcal{M}$ with a tracial state τ. Let $\delta_0(x_1, \ldots, x_n)$ be Voiculescu's free entropy dimension. Then

$$\delta_0(x_1, \ldots, x_n) \leq \mathfrak{K}_2(x_1, \ldots, x_n) + 1.$$

Theorem 3.4 (Hadwin & Shen (2007))

Suppose $\mathcal{M}$ is a von Neumann algebra with a tracial state τ and is generated by a family of elements $\{x_1, \ldots, x_n\}$ as a von Neumann algebra. If

$$\mathfrak{K}_2(x_1, \ldots, x_n) = 0,$$

then

$$\mathfrak{K}_2(\mathcal{M}) = 0.$$

In particular,

$$\mathcal{M} \not\simeq L(F(n)), \qquad \text{for } n \geq 2.$$

Theorem 3.5 (Hadwin & Shen (2007))

If $\mathcal{M}$ is a hyperfinite von Neumann algebra with a tracial state τ, then $\mathfrak{K}_2(\mathcal{M}) = 0$.

Theorem 3.6 (Hadwin & Shen (2007))

If $\mathfrak{K}_2(\mathcal{N}_1) = \mathfrak{K}_2(\mathcal{N}_2) = 0$ and $\mathcal{N}_1 \cap \mathcal{N}_2$ is diffuse, then $\mathfrak{K}_2((\mathcal{N}_1 \cup \mathcal{N}_2)'') = 0$.

Theorem 3.7 (Hadwin & Shen (2007))

If $\mathcal{M} = \{\mathcal{N}, u\}''$ where $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N}) = 0$ and u is a unitary element in $\mathcal{M}$ satisfying, for a sequence $\{v_n\}$ of Haar unitary elements in $\mathcal{N}$, $dist_{\|\|_2}(uv_nu^*, \mathcal{N}) \to 0$, then $\mathfrak{K}_2(\mathcal{M}) = 0$.

Theorem 3.8 (Hadwin & Shen (2007))

If $\{\mathcal{N}_i\}_{i=1}^{\infty}$ is an ascending sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\mathfrak{K}_2(\mathcal{N}_i) = 0$ for all $i \geq 1$ and $\mathcal{M} = \overline{\cup_i \mathcal{N}_i}^{SOT}$, then $\mathfrak{K}_2(\mathcal{M}) = 0$.

Theorem 3.9 (Hadwin & Shen (2007))

Let $\mathcal{M}$ is a II_1 factor with trace state τ. Assume that $\mathcal{N}_1, \mathcal{N}_2$ are von Neumann subalgebra of $\mathcal{M}$ with $\mathfrak{K}_2(\mathcal{N}_1) = \mathfrak{K}_2(\mathcal{N}_2) = 0$ and $\xi_1, \ldots \xi_n \in L^2(\mathcal{M}, \tau)$ with

$$\overline{span}^{\|\cdot\|_2}(\mathcal{N}_1\{x_1, \ldots, \xi_n\}\mathcal{N}_2) = L^2(\mathcal{M}, \tau).$$

Then $\delta_0(M) \leq 2 + 2n$.

Definition 3.10 (Hadwin, Li, & Shen (2011))

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful tracial state τ. Assume $x_1, \ldots, x_n$ are in $\mathcal{M}$. Let

$$\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon; \tau)$$

be defined as previous. We define

$$\begin{aligned}
&\mathfrak{K}_2^{(2)}(x_1, , \ldots, x_n; \omega; \tau) \\
&\qquad = \sup_{R>0} \inf_{m\in\mathbb{N}, \epsilon>0} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R(x_1, \ldots, x_n; m, k, \epsilon; \tau), \omega))}{k^2} \\
&\mathfrak{K}_2^{(2)}(x_1, , \ldots, x_n; \tau) = \limsup_{\omega\to 0^+} \mathfrak{K}_2^{(2)}(x_1, , \ldots, x_n; \omega; \tau),
\end{aligned}$$

where $\mathfrak{K}_2^{(2)}(x_1, \ldots, x_n; \tau)$ is called the modified free orbit-dimension *of $x_1, \ldots, x_n$ with respect to the tracial state τ.*

The concept of “strongly 1-bounded von Neumann algebra” was independently introduced by K. Jung. Though the definitions of “free orbit diemsnion” and “strongly 1-bounded” are different, they share many similar properties.

Voiculescu's Topological Free Entropy Theory

Assume that $\mathcal{A}$ is a unital C*-algebra. Let $x_1, \ldots, x_n$ be self-adjoint elements in $\mathcal{A}$. Let $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ be the unital noncommutative polynomials in the indeterminates $X_1, \ldots, X_n$. Let $\{P_r\}_{r=1}^{\infty}$ be the collection of all noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ with rational complex coefficients.

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$. For every $\omega > 0$, we define the ω-$\|\cdot\|$-ball $Ball(B_1, \ldots, B_n; \omega, \|\cdot\|)$ centered at $(B_1, \ldots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\| < \omega.$$

Definition 4.1

Suppose that Σ is a set in $\mathcal{M}_k(\mathbb{C})^n$. We define $\nu_\infty(\Sigma, \omega)$ to be the minimal number of ω-$\|\cdot\|$-balls that consist a covering of Σ in $\mathcal{M}_k(\mathbb{C})^n$.

For all integers $r, k \geq 1$, real numbers $R, \epsilon > 0$ and noncommutative polynomials $P_1, \ldots, P_r$, we define

$$\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r)$$

to be the subset of $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ consisting of all these $(A_1, \ldots, A_n) \in (\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ satisfying $\max\{\|A_1\|, \ldots, \|A_n\|\} \leq R$ and $|\|P_j(A_1, \ldots, A_n)\| - \|P_j(x_1, \ldots, x_n)\|| \leq \epsilon, \qquad \forall\, 1 \leq j \leq r$.
Define

$$\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set

$$\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r)$$

by balls with radius ω in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with operator norm.

Define

$$\delta_{top}(x_1, \ldots, x_n; \omega) = \sup_{R>0} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(\nu_\infty(\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \omega))}{-k^2 \log \omega}$$

The topological free entropy dimension of $x_1, \ldots, x_n$ is defined by

$$\delta_{top}(x_1, \ldots, x_n) = \limsup_{\omega\to 0^+} \delta_{top}(x_1, \ldots, x_n; \omega)$$

Voiculescu discussed many important properties of topological free entropy dimension, some of which are listed as follows.

Lemma 4.2 (Voiculescu)

(i) If $x_1, \ldots, x_n$ is a family of free semicircular elements in a unital C^* algebra with a tracial state, then

$$\delta_{top}(x_1, \ldots, x_n) = n.$$

(ii) If $x_1, \ldots, x_n$ is the universal n-tuple of self-adjoint contractions, then

$$\delta_{top}(x_1, \ldots, x_n) = n.$$

One variable case

We study the topological free entropy dimension of a single self-adjoint element.

Theorem 4.3 (Hadwin & Shen (2009))

Suppose that x is a self-adjoint element in a unital C^*-algebra $\mathcal{A}$. Then

$$\delta_{top}(x) = 1 - \frac{1}{n};$$

where n is the cardinality of the set of the spetrum of x in $\mathcal{A}$.

Finite dimensional C*-algebras In the case of finite dimensional C*-algebra, we obtain

Theorem 4.4

Suppose that $\mathcal{A}$ is a finite dimensional C^*-algebra with a family of self-adjoint generators $x_1, \ldots, x_n$. Then

$$\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{dim_{\mathbb{C}}\mathcal{A}},$$

where $dim_{\mathbb{C}}\mathcal{A}$ is the complex dimension of the C^*-algebra $\mathcal{A}$.

Topological orbit dimension $\mathfrak{K}_{top}^{(2)}$

Define

$$o_2(\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \omega)$$

to be the covering number of the set

$$\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r)$$

by ω-orbit-$\|\cdot\|_2$-balls in the metric space $(\mathcal{M}_k^{s.a}(\mathbb{C}))^n$ equipped with the trace norm.

Definition 4.5 (Hadwin, Li & Shen (2011))

Define

$$\mathfrak{K}_{top}^{(2)}(x_1, \ldots, x_n; \omega) = \sup_{R>0} \inf_{\epsilon>0, r\in\mathbb{N}} \limsup_{k\to\infty} \frac{\log(o_2(\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r), \omega))}{k^2}$$

Definition 4.6

The topological orbit dimension *of* $x_1, \ldots, x_n$ *is defined by*

$$\mathfrak{K}_{top}^{(2)}(x_1, \ldots, \ x_n) = \limsup_{\omega \to 0^+} \mathfrak{K}_{top}^{(2)}(x_1, \ldots, x_n; \omega)$$

Remark 4.7

In the notation $\mathfrak{K}_{top}^{(2)}$*, the subscript “top" stands for the norm-microstates space and superscript “(2)" stands for the using of unitary-orbit-*$\|\cdot\|_2$*-balls when counting the covering numbers of the norm-microstates spaces.*

Theorem 4.8 (Hadwin, Li & Shen (2011))

Suppose that $\mathcal{A}$ is a unital C^*-algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Then

$$\delta_{top}(x_1, \ldots, x_n) \leq \max\{\mathfrak{K}_{top}^{(2)}(x_1, \ldots, x_n), 1\}.$$

In particular, if

$$\mathfrak{K}_{top}^{(2)}(x_1, \ldots, x_n) = 0,$$

then

$$\delta_{top}(x_1, \ldots, x_n) \leq 1.$$

Theorem 4.9 (Hadwin, Li & Shen (2011))

Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_p\}$ are two families of self-adjoint generators of $\mathcal{A}$. Then

$$\mathfrak{K}_{top}^{(2)}(x_1, \ldots, x_n) = \mathfrak{K}_{top}^{(2)}(y_1, \ldots, y_p).$$

Theorem 4.10 (Hadwin, Li & Shen (2011))

Suppose that $\mathcal{A}$ is a unital C^*-algebra and $x_1, \ldots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$. Then

$$\mathfrak{K}_{top}^{(2)}(x_1, \ldots, x_n) \leq \sup_{\tau \in TS(\mathcal{A})} \mathfrak{K}_2^{(2)}(x_1, \ldots, x_n; \tau)$$

The proof of the result follows the ideas developed by Voicuelescu in his proof that

$$\chi_{top}(x_1, \ldots, x_n) \leq \sup_{\tau \in TS(A)} \chi(x_1, \ldots, x_n; \tau).$$

In the case of nuclear C*-algebra, we proved the following result.

Theorem 4.11 (Hadwin, Li & Shen (2011))

Suppose $\mathcal{A}$ is a finitely generated nuclear C*-algebra with a family of self-adjoint generators $x_1, \ldots, x_n$. Then

$$\delta_{top}(x_1, \ldots, x_n) \leq 1.$$

Example 4.12

Let $\mathcal{A} = \mathcal{K} + \mathbb{C}I$ where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space. Then

$$\delta_{top}(x_1, \ldots, x_n) = 0, \text{ for each family of generators } x_1, \ldots, x_n \text{ in } \mathcal{A}.$$

MF algebras

A separable C*-algebra $\mathcal{A}$ is an MF-algebra if $\mathcal{A}$ can be embedded into $\prod_{k\geq 1} M_{m_k}(\mathbb{C})/\sum_{k\geq 1} M_{m_k}(\mathbb{C})$ for some increasing sequence $\{m_k\}$ of positive integers.
Let $x_1, \ldots, x_n$ be a family of self-adjoint elements in $\mathcal{A}$. The C*-algebra $C^*(x_1, \ldots, x_n)$ is an MF-algebras if and only if there are an increasing sequence $\{m_k\}$ of positive integers and sequences $\{A_{1,k}\}, \ldots, \{A_{nk}\}$ with $A_{1,k}, \ldots, A_{n,k}$ in $M_{m_k}(\mathbb{C})$ such that

$$\lim_{k\to\infty} \|Q(A_{1,k}, \ldots, A_{n,k})\| = \|Q(x_1, \ldots, x_n)\|$$

for every polynomial $Q(t_1, \ldots, t_n)$. I.e., for sufficiently large k,

$$\Gamma_R^{(top)}(x_1, \ldots, x_n; k, \epsilon, P_1, \ldots, P_r)$$

is nonempty.

MF trace

Definition 4.13

Suppose that $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an MF-algebra. A tracial state τ on $\mathcal{A}$ is an MF-trace if there are an increasing sequence $\{m_k\}$ of positive integers and sequences $\{A_{1,k}\}, \ldots, \{A_{nk}\}$ with $A_{1,k}, \ldots, A_{n,k}$ in $M_{m_k}(\mathbb{C})$ such that

- $\lim_{k\to\infty} \|Q(A_{1,k}, \ldots, A_{n,k})\| = \|Q(x_1, \ldots, x_n)\|$ *for every polynomial* $Q(t_1, \ldots, t_n)$.
- $\lim_{k\to\infty} \tau_{m_k}(Q(A_{1,k}, \ldots, A_{n,k})) = \tau(\|Q(x_1, \ldots, x_n))$ *for every polynomial* $Q(t_1, \ldots, t_n)$.

We let $\mathcal{T}_{MF}(\mathcal{A})$ denote the set of all MF-traces of $\mathcal{A}$.

Upper bound of topological free entropy

Theorem 4.14 (Hadwin, Li, Li & Shen (2020))

Let $\mathcal{A}$ be a unital C^*-algebra and $x_1, \ldots, x_n$ be a family of self-adjoint generating elements of $\mathcal{A}$. Then

$$\mathcal{K}_{top}^{(2)}(x_1, \ldots, x_n) \leq \sup_{\tau \in \mathcal{T}_{MF}(\mathcal{A})} \mathcal{K}_2^{(2)}(x_1, \ldots, x_n).$$

The proof of the result follows the ideas developed by Voicuelescu in his proof that

$$\chi_{top}(x_1, \ldots, x_n) \leq \sup_{\tau \in TS(A)} \chi(x_1, \ldots, x_n; \tau).$$

Lower bound of topological free entropy

Theorem 4.15 (Hadwin, Li, Li & Shen (2014))

Let $\mathcal{A}$ be a unital C^*-algebra and $x_1, \ldots, x_n$ be a family of self-adjoint generating elements of $\mathcal{A}$. If $\mathcal{A}$ is MF-nuclear, then

$$\delta_{top}(x_1, \ldots, x_n) \geq \sup_{\tau \in \mathcal{T}_{MF}(\mathcal{A})} \delta_0(x_1, \ldots, x_n; \tau).$$

Recall that $\mathcal{A}$ is MF-nuclear if $\pi_\tau(\mathcal{A})''$ is hyperfinite for each $\tau \in \mathcal{T}_{MF}(\mathcal{A})$, where π_τ is the GNS representation of $\mathcal{A}$ associated with τ.

Theorem 4.16 (Hadwin, Li, Li & Shen (2014))

Let $\mathcal{A}$ be a unital C^*-algebra and $x_1, \ldots, x_n$ be a family of self-adjoint generating elements of $\mathcal{A}$. If $\mathcal{A}$ is an MF algebra and $\tau \in \mathcal{T}_{MF}(\mathcal{A})$. Then

$$\delta_{top}(x_1, \ldots, x_n) \geq \delta_0(b; \tau), \qquad \forall\, b \in \pi_\tau(\mathcal{A})'',$$

where π_τ is the GNS-representation of $\mathcal{A}$ associated with τ.

Some classes of C*-algebras

Definition 4.17

A separable C-algebra is inner QD if and only if it has a separating family of quasidiagonal irreducible representations.*

Theorem 4.18 (Hadwin, Li, Li & Shen (2020))

Suppose that $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an MF-algebra and inner quasidiagonal, then $\delta_{top}(x_1, \ldots, x_n) = 1 - \frac{1}{dim(\mathcal{A})}$.

Theorem 4.19 (Hadwin, Li, Li & Shen (2020))

Suppose that $\mathcal{A} = C^*(x_1, \ldots, x_n)$ is an MF-algebra with no finite-dimensional representations. If $\mathcal{A}$ has property c-Γ, then $\delta_{top}(x_1, \ldots, x_n) = 1$.*

A separpable C*-algebra $\mathcal{A}$ has property c*-Γ, if $\pi(\mathcal{A})''$ is a II_1 factor of Property Γ for each II_1 factor representation π of $\mathcal{A}$.

Free orbit-dimension for non-separable C*-algebras

Suppose that $\mathcal{A}$ is a unital C*-algbera. Let $\mathcal{G} \subseteq \mathcal{A}$. Define

$$\mathfrak{K}^{(3)}_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) = \begin{cases} 0 \text{ if } \mathfrak{K}^{(2)}_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) = 0 \\ \infty \text{ if } \mathfrak{K}^{(2)}_{top}(x_1, \ldots, x_n : y_1, \ldots, y_m) > 0 \end{cases}$$

$$\mathfrak{K}^{(3)}_{top}(\mathcal{G}) = \sup_{\text{finite } E \subseteq \mathcal{G}} \inf_{\text{finite } F \subseteq \mathcal{G}} \mathfrak{K}^{(3)}_{top}(E : F)$$

Many similar results can be obtained.

Thank you!