Bi-free probability and reflection positivity

Roland Speicher

Saarland University
Saarbrücken, Germany

arXiv: 2312.06813

Historical background

Mathematisches Forschungsinstitut Oberwolfach

Report No. 55/2017

DOI: 10.4171/OWR/2017/55

Reflection Positivity

Organised by
Arthur Jaffe, Harvard
Karl-Hermann Neeb, Erlangen
Gestur Olafsson, Baton Rouge
Benjamin Schlein, Zürich

26 November – 2 December 2017

Historical background

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 55/2017

DOI: 10.4171/OWR/2017/55

Reflection Positivity

Organised by
Arthur Jaffe, Harvard
Karl-Hermann Neeb, Erlangen
Gestur Olafsson, Baton Rouge
Benjamin Schlein, Zürich

26 November – 2 December 2017

Note on
Bi-free probability and reflection positivity

Historical background

Obituary: K.R. Parthasarathy 1936–2023

JULY 16, 2023

Kalyanapuram Rangachari Parthasarathy, known to generations of mathematicians and probabilists simply as **KRP**, passed away on June 14 in New Delhi; he was 86. Professor Parthasarathy made numerous extremely deep contributions over a stunningly wide spectrum of mathematics: probability, quantum probability, graph theory, linear algebra, statistics and other mathematical domains. With his passing, India has lost an icon of 20th century mathematics.

Announcement:

Special Issue of IDAQP in honour of Prof K R Parthasarathy

Historical background

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 55/2017

DOI: 10.4171/OWR/2017/55

Reflection Positivity

Organised by
Arthur Jaffe, Harvard
Karl-Hermann Neeb, Erlangen
Gestur Olafsson, Baton Rouge
Benjamin Schlein, Zürich

26 November – 2 December 2017

Note on
Bi-free probability and reflection positivity

Obituary: K.R. Parthasarathy 1936–2023

JULY 16, 2023

Kalyanapuram Rangachari Parthasarathy, known to generations of mathematicians and probabilists simply as **KRP**, passed away on June 14 in New Delhi; he was 86. Professor Parthasarathy made numerous extremely deep contributions over a stunningly wide spectrum of mathematics: probability, quantum probability, graph theory, linear algebra, statistics and other mathematical domains. With his passing, India has lost an icon of 20th century mathematics.

Announcement:

Special Issue of IDAQP in honour of Prof K R Parthasarathy

Historical background

arXiv:2312.06813

BI-FREE PROBABILITY THEORY AND REFLECTION POSITIVITY

ROLAND SPEICHER

ABSTRACT. We point out that bi-free product constructions respect reflection positivity.

Obituary: K.R. Parthasarathy 1936–2023

JULY 16, 2023

Kalyanapuram Rangachari Parthasarathy, known to generations of mathematicians and probabilists simply as **KRP**, passed away on June 14 in New Delhi; he was 86. Professor Parthasarathy made numerous extremely deep contributions over a stunningly wide spectrum of mathematics: probability, quantum probability, graph theory, linear algebra, statistics and other mathematical domains. With his passing, India has lost an icon of 20th century mathematics.

Announcement:

Special Issue of IDAQP in honour of Prof K R Parthasarathy

Historical background

arXiv:2312.06813

BI-FREE PROBABILITY THEORY AND REFLECTION POSITIVITY

ROLAND SPEICHER

ABSTRACT. We point out that bi-free product constructions respect reflection positivity.

Probabilistic Operator Algebra Seminar

Organizer: Dan-Virgil Voiculescu

January 30 **Roland Speicher**, Saarland University Saarbruecken

Title: *Bi-free probability and reflection positivity*

I will recall the notions of reflection positivity (from algebraic quantum field theory) and point out that bi-free product constructions respect reflection positivity.

Obituary: K.R. Parthasarathy 1936–2023

JULY 16, 2023

Kalyanapuram Rangachari Parthasarathy, known to generations of mathematicians and probabilists simply as **KRP**, passed away on June 14 in New Delhi; he was 86. Professor Parthasarathy made numerous extremely deep contributions over a stunningly wide spectrum of mathematics: probability, quantum probability, graph theory, linear algebra, statistics and other mathematical domains. With his passing, India has lost an icon of 20th century mathematics.

Announcement:

Special Issue of IDAQP in honour of Prof K R Parthasarathy

What is reflection positivity?

Reflection positivity is . . .

. . . some additional positivity, corresponding to a symmetry θ, in addition to the "usual" positivity.

What is reflection positivity?

Hilbert space setting

- Hilbert space $\mathcal{H}, \langle \cdot, \cdot \rangle$
- unitary involution $\theta : \mathcal{H} \to \mathcal{H}$, i.e., $\theta^2 = \text{id}$ and $\langle \theta(f), \theta(g) \rangle = \langle f, g \rangle$
- distinguished subspace $\mathcal{H}_+ \subset \mathcal{H}$,

$(\mathcal{H}, \mathcal{H}_+, \theta)$ is called *reflection positive* if

$$\langle \theta(f), f \rangle \geq 0 \qquad \text{for all } f \in \mathcal{H}_+$$

What is reflection positivity?

Hilbert space setting

- Hilbert space $\mathcal{H}, \langle \cdot, \cdot \rangle$
- unitary involution $\theta : \mathcal{H} \to \mathcal{H}$, i.e., $\theta^2 = \mathrm{id}$ and $\langle \theta(f), \theta(g) \rangle = \langle f, g \rangle$
- distinguished subspace $\mathcal{H}_+ \subset \mathcal{H}$, often $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathcal{H}_- = \theta(\mathcal{H}_+)$

$(\mathcal{H}, \mathcal{H}_+, \theta)$ is called *reflection positive* if

$$\langle \theta(f), f \rangle \geq 0 \qquad \text{for all } f \in \mathcal{H}_+$$

What is reflection positivity?

Hilbert space setting

$(\mathcal{H}, \mathcal{H}_+, \theta)$ is called *reflection positive* if

$$\langle \theta(f), f \rangle \geq 0 \qquad \text{for all } f \in \mathcal{H}_+$$

Operator algebra setting

- operator algebra A with state $\tau : A \to \mathbb{C}$
- involution $\theta : A \to A$, anti-linear homomorphism, $\tau \circ \theta = \tau$
- distinguished subalgebra $A_+ \subset A$

(A, A_+, θ) is called *reflection positive* if

$$\tau(\theta(x)x) \geq 0 \qquad \text{for all } x \in A_+$$

What is reflection positivity?

Hilbert space setting

$(\mathcal{H}, \mathcal{H}_+, \theta)$ is called *reflection positive* if

$$\langle \theta(f), f \rangle \geq 0 \qquad \text{for all } f \in \mathcal{H}_+$$

Operator algebra setting

- operator algebra A with state $\tau : A \to \mathbb{C}$
- involution $\theta : A \to A$, anti-linear homomorphism, $\tau \circ \theta = \tau$
- distinguished subalgebra $A_+ \subset A$

(A, A_+, θ) is called *reflection positive* if

$$\tau(\theta(x)x) \geq 0 \qquad \text{for all } x \in A_+$$

Often one also requires that A_+ and $A_- := \theta(A_+)$ commute (or satisfy some other relations) and both together generate A

What is reflection positivity?

Hilbert space setting

$(\mathcal{H}, \mathcal{H}_+, \theta)$ is called *reflection positive* if

$$\langle \theta(f), f \rangle \geq 0 \qquad \text{for all } f \in \mathcal{H}_+$$

Operator algebra setting

(A, A_+, θ) is called *reflection positive* if

$$\tau(\theta(x)x) \geq 0 \qquad \text{for all } x \in A_+$$

Reflection positivity is ...

... some additional positivity, corresponding to a symmetry θ, in addition to the "usual" positivity.

Typical examples for symmetries θ

- reflection $t \to -t$

$$\theta(f)(t) = f(-t)$$

Typical examples for symmetries θ

- reflection $t \to -t$

$$\theta(f)(t) = f(-t)$$

- left-right exchange between algebra and commutant for von Neumann algebras M

$$A_+ := M, \qquad A_- := M'$$

$$\theta : M \to M', \qquad a \mapsto \theta(a) = JaJ,$$

Typical examples for symmetries θ

- reflection $t \to -t$

$$\theta(f)(t) = f(-t)$$

- left-right exchange between algebra and commutant for von Neumann algebras M

$$A_+ := M, \qquad A_- := M'$$

$$\theta : M \to M', \qquad a \mapsto \theta(a) = JaJ,$$

$$\tau(\theta(a)a) = \langle \Omega, JaJa\Omega \rangle = \langle Ja^*\Omega, a\Omega \rangle = \langle \Delta^{1/2} a\Omega, a\Omega \rangle \geq 0$$

Typical examples for symmetries θ

- reflection $t \to -t$

$$\theta(f)(t) = f(-t)$$

- left-right exchange between algebra and commutant for von Neumann algebras M

$$A_+ := M, \qquad A_- := M'$$

$$\theta : M \to M', \qquad a \mapsto \theta(a) = JaJ,$$

$$\tau(\theta(a)a) = \langle \Omega, JaJa\Omega \rangle = \langle Ja^*\Omega, a\Omega \rangle = \langle \Delta^{1/2} a\Omega, a\Omega \rangle \geq 0$$

- left-right exchange between two faces in bi-free setting

$(x_i, y_i)_{i \in I}$ pairs of faces and $\theta(x_i) = y_i$

Osterwalder-Schrader axioms for euclidian QFT

Commun. math. Phys. 31, 83–112 (1973)

Axioms for Euclidean Green's Functions

Konrad Osterwalder* and Robert Schrader**

Lyman Laboratory of Physics, Harvard University, Cambridge, Mass. USA

Received December 18, 1972

Abstract. We establish necessary and sufficient conditions for Euclidean Green's functions to define a unique Wightman field theory.

Contents

relativistic/Minkowski QFT

Euclidean QFT

relativistic/Minkowski QFT

- Wightmann functions

Euclidean QFT

- Schwinger functions

relativistic/Minkowski QFT

- Wightmann functions
- $\psi(x,t)$

Euclidean QFT

- Schwinger functions
- $\tilde{\psi}(x,\tau) = \psi(x, t = i\tau)$

relativistic/Minkowski QFT

- Wightmann functions
- $\psi(x,t)$
- $\psi(x,t) = e^{itH}\psi(x,0)e^{-itH}$

Euclidean QFT

- Schwinger functions
- $\tilde{\psi}(x,\tau) = \psi(x, t = i\tau)$
- $\tilde{\psi}(x,\tau) = e^{-\tau H}\psi(x,0)e^{\tau H}$

relativistic/Minkowski QFT

- Wightmann functions
- $\psi(x,t)$
- $\psi(x,t) = e^{itH}\psi(x,0)e^{-itH}$
- $\psi(x,t)^* = \psi(x,t)$

Euclidean QFT

- Schwinger functions
- $\tilde{\psi}(x,\tau) = \psi(x, t = i\tau)$
- $\tilde{\psi}(x,\tau) = e^{-\tau H}\psi(x,0)e^{\tau H}$
- $\widetilde{\psi}(x,\tau)^* = e^{\tau H}\psi(x,0)e^{-\tau H} = \widetilde{\psi}(x,-\tau)$

relativistic/Minkowski QFT

- Wightmann functions
- $\psi(x,t)$
- $\psi(x,t) = e^{itH}\psi(x,0)e^{-itH}$
- $\psi(x,t)^* = \psi(x,t)$

Euclidean QFT

- Schwinger functions
- $\tilde{\psi}(x,\tau) = \psi(x, t = i\tau)$
- $\tilde{\psi}(x,\tau) = e^{-\tau H}\psi(x,0)e^{\tau H}$
- $\widetilde{\psi}(x,\tau)^* = e^{\tau H}\psi(x,0)e^{-\tau H} = \widetilde{\psi}(x,-\tau)$
- And thus, with $\theta\big(\widetilde{\psi}\big)(x,\tau) = \widetilde{\psi}(x,-\tau)$,

$$\tau[\theta(\tilde{\psi})\tilde{\psi}] = \tau[\tilde{\psi}^*\tilde{\psi}] \geq 0$$

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int \int \bar{f}(s) g(t) k(s,t) ds dt$$

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int \int \bar{f}(s) g(t) k(s,t) ds dt$$

positivity of inner product: $k(\cdot, \cdot)$ is positive definite kernel

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int\int \bar{f}(s) g(t) k(s,t) ds dt$$

$$\theta : \mathcal{H} \to \mathcal{H}; \qquad \theta(f)(t) = f(-t)$$

$\mathcal{H}_+ := \{f \in L^2 | f(t) = 0 \text{ for all } t \in \mathbb{R}_-\} \equiv L^2(\mathbb{R}_+, k)$,

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int\int \bar{f}(s) g(t) k(s,t) ds dt$$

$$\theta : \mathcal{H} \to \mathcal{H}; \qquad \theta(f)(t) = f(-t)$$

$$\mathcal{H}_+ := L^2(\mathbb{R}_+, k), \qquad \theta(\mathcal{H}_+) = \mathcal{H}_- \equiv L^2(\mathbb{R}_-, k)$$

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int \int \bar{f}(s) g(t) k(s,t) ds dt$$

$$\theta : \mathcal{H} \to \mathcal{H}; \qquad \theta(f)(t) = f(-t)$$

$$\mathcal{H}_+ := L^2(\mathbb{R}_+, k), \qquad \theta(\mathcal{H}_+) = \mathcal{H}_- \equiv L^2(\mathbb{R}_-, k)$$

$$\begin{aligned} f \in \mathcal{H}_+ : \quad \langle \theta(f), f \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\theta(f)}(s) f(t) k(s,t) ds dt \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \bar{f}(-s) f(t) k(s,t) ds dt \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \bar{f}(s) f(t) k(-s,t) ds dt \end{aligned}$$

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int \int \bar{f}(s) g(t) k(s,t) ds dt$$

$$\theta : \mathcal{H} \to \mathcal{H}; \qquad \theta(f)(t) = f(-t)$$

$$\mathcal{H}_+ := L^2(\mathbb{R}_+, k), \qquad \theta(\mathcal{H}_+) = \mathcal{H}_- \equiv L^2(\mathbb{R}_-, k)$$

$$f \in \mathcal{H}_+ : \quad \langle \theta(f), f \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \bar{f}(s) f(t) k(-s,t) ds dt$$

Meaning of reflection positivity

$$\mathcal{H} = L^2(\mathbb{R}, k), \qquad \langle f, g \rangle = \int \int \bar{f}(s) g(t) k(s,t) ds dt$$

$$\theta : \mathcal{H} \to \mathcal{H}; \qquad \theta(f)(t) = f(-t)$$

$$\mathcal{H}_+ := L^2(\mathbb{R}_+, k), \qquad \theta(\mathcal{H}_+) = \mathcal{H}_- \equiv L^2(\mathbb{R}_-, k)$$

$$f \in \mathcal{H}_+ : \quad \langle \theta(f), f \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \bar{f}(s) f(t) k(-s,t) ds dt$$

Reflection positive kernels (Jorgensen, Neeb, Olafson)

We have on $L^2(\mathbb{R}_+)$ two positve definite kernels, given via k, namely

- $(s,t) \mapsto k(s,t)$ or $(s,t) \mapsto k(t-s)$
- $(s,t) \mapsto k(-s,t)$ or $(s,t) \mapsto k(t+s)$

Consequences of reflection positivity

What can we get out of reflection positivity?

Consequences of reflection positivity

What can we get out of reflection positivity?

$$\langle f, g\rangle_\theta := \langle \theta(f), g\rangle \qquad \text{for } f, g \in \mathcal{H}_+$$

gives a new inner product on $\mathcal{H}_+$, and thus we have Cauchy-Schwartz also for this

$$\langle f, g\rangle_\theta^2 \leq \langle f, f\rangle_\theta \cdot \langle g, g\rangle_\theta,$$

Consequences of reflection positivity

What can we get out of reflection positivity?

$$\langle f, g \rangle_\theta := \langle \theta(f), g \rangle \qquad \text{for } f, g \in \mathcal{H}_+$$

gives a new inner product on $\mathcal{H}_+$, and thus we have Cauchy-Schwartz also for this

$$\langle f, g \rangle_\theta^2 \leq \langle f, f \rangle_\theta \cdot \langle g, g \rangle_\theta,$$

or in terms of the old inner product

$$\langle \theta(f), g \rangle \leq \sqrt{\langle \theta(f), f \rangle \cdot \langle \theta(g), g \rangle} \leq \frac{1}{2}\big\{\langle \theta(f), f \rangle + \langle \theta(g), g \rangle\big\}$$

Consequences of reflection positivity

What can we get out of reflection positivity?

$$\langle f, g\rangle_\theta := \langle \theta(f), g\rangle \qquad \text{for } f, g \in \mathcal{H}_+$$

gives a new inner product on $\mathcal{H}_+$, and thus we have Cauchy-Schwartz also for this

$$\langle f, g\rangle_\theta^2 \leq \langle f, f\rangle_\theta \cdot \langle g, g\rangle_\theta,$$

or in terms of the old inner product

$$\langle \theta(f), g\rangle \leq \sqrt{\langle \theta(f), f\rangle \cdot \langle \theta(g), g\rangle} \leq \frac{1}{2}\big\{\langle \theta(f), f\rangle + \langle \theta(g), g\rangle\big\}$$

and so

$$\begin{aligned}\langle \theta(f) + g, \theta(f) + g\rangle &= \langle \theta(f), \theta(f)\rangle + \langle g, g\rangle + 2\langle \theta(f), g\rangle \\ &\leq \langle \theta(f), \theta(f)\rangle + \langle g, g\rangle + \langle \theta(f), f\rangle + \langle \theta(g), g\rangle\} \\ &= \frac{1}{2}\big\{\langle f + \theta(f), f + \theta(f)\rangle + \langle g + \theta(g), g + \theta(g)\rangle\big\}\end{aligned}$$

Consequences of reflection positivity

$$\langle \theta(f) + g, \theta(f) + g \rangle = \frac{1}{2}\big\{\langle f + \theta(f), f + \theta(f) \rangle + \langle g + \theta(g), g + \theta(g) \rangle\big\}$$

Consequences of reflection positivity

$$\langle\theta(f)+g,\theta(f)+g\rangle = \frac{1}{2}\big\{\langle f+\theta(f), f+\theta(f)\rangle + \langle g+\theta(g), g+\theta(g)\rangle\big\}$$

Say, we want to maximize

$$\mathcal{E}(h) = \langle h,h\rangle = \int\int \bar{h}(s)h(t)k(s,t)dsdt$$

Consequences of reflection positivity

$$\langle\theta(f)+g,\theta(f)+g\rangle=\frac{1}{2}\big\{\langle f+\theta(f),f+\theta(f)\rangle+\langle g+\theta(g),g+\theta(g)\rangle\big\}$$

Say, we want to maximize

$$\mathcal{E}(h)=\langle h,h\rangle=\int\int\bar{h}(s)h(t)k(s,t)dsdt$$

Write

$$h=h_-+h_+,\qquad \text{with } h_+\in L^2(\mathbb{R}_+) \text{ and } h_-\in L^2(\mathbb{R}_-)$$

then

$$\mathcal{E}(h)=\mathcal{E}(h_-+h_+)\le\frac{\mathcal{E}\big(\theta(h_-)+h_-\big)+\mathcal{E}\big(h_++\theta(h_+)\big)}{2}$$

Consequences of reflection positivity

$$\mathcal{E}(h) = \langle h, h \rangle = \int \int \bar{h}(s) h(t) k(s,t) ds dt$$

$$\mathcal{E}(h) = \mathcal{E}(h_- + h_+) \leq \frac{\mathcal{E}\big(\theta(h_-) + h_-\big) + \mathcal{E}\big(h_+ + \theta(h_+)\big)}{2}$$

Consequences of reflection positivity

$$\mathcal{E}(h) = \langle h, h \rangle = \int \int \bar{h}(s)h(t)k(s,t)dsdt$$

$$\mathcal{E}(h) = \mathcal{E}(h_- + h_+) \leq \frac{\mathcal{E}\big(\theta(h_-) + h_-\big) + \mathcal{E}\big(h_+ + \theta(h_+)\big)}{2}$$

Symmetric problem has symmetric solution

Thus, at least one of $h_- + \theta(h_-)$ and $h_+ + \theta(h_+)$ is a better maximizer for $\mathcal{E}$ than $h = h_- + h_+$. This means that the solution h to the maximization problem must be symmetric, i.e., $\theta(h) = h$.

Consequences of reflection positivity

$$\mathcal{E}(h) = \langle h, h \rangle = \int\int \bar{h}(s)h(t)k(s,t)dsdt$$

$$\mathcal{E}(h) = \mathcal{E}(h_- + h_+) \leq \frac{\mathcal{E}\big(\theta(h_-) + h_-\big) + \mathcal{E}\big(h_+ + \theta(h_+)\big)}{2}$$

Symmetric problem has symmetric solution

Thus, at least one of $h_- + \theta(h_-)$ and $h_+ + \theta(h_+)$ is a better maximizer for $\mathcal{E}$ than $h = h_- + h_+$. This means that the solution h to the maximization problem must be symmetric, i.e., $\theta(h) = h$.

Existence of phase transitions in statistical physics (Fröhlich, Israel, Lieb, Simon)

- symmetry of solution for small temperatures by reflection positivity
- absence of such symmetry by high temperature expansion

Meaning of reflection positivity

Consider two commuting random variables x and y

$$A = L^\infty(\mathbb{R} \times \mathbb{R}) = L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R}) = \{f(x,y)\}$$

$$\theta(f)(x,y) = \bar{f}(y,x)$$

$$A_+ = L^\infty(\mathbb{R}) \otimes 1 = \{f(x)\}, \qquad A_- = 1 \otimes L^\infty(\mathbb{R}) = \{f(y)\}$$

Meaning of reflection positivity

Consider two commuting random variables x and y

$$A = L^\infty(\mathbb{R} \times \mathbb{R}) = L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R}) = \{f(x,y)\}$$

$$\tau[f(x,y] = \int\int f(s,t)k(s,t)dsdt$$

$$\theta(f)(x,y) = \bar{f}(y,x)$$

$$A_+ = L^\infty(\mathbb{R}) \otimes 1 = \{f(x)\}, \qquad A_- = 1 \otimes L^\infty(\mathbb{R}) = \{f(y)\}$$

Meaning of reflection positivity

Consider two commuting random variables x and y

Positivity: $k(\cdot,\cdot)$ positive density

$$A = L^\infty(\mathbb{R}\times\mathbb{R}) = L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R}) = \{f(x,y)\}$$

$$\tau[f(x,y] = \int\int f(s,t)k(s,t)dsdt$$

$$\theta(f)(x,y) = \bar{f}(y,x)$$

$$A_+ = L^\infty(\mathbb{R})\otimes 1 = \{f(x)\}, \qquad A_- = 1\otimes L^\infty(\mathbb{R}) = \{f(y)\}$$

Meaning of reflection positivity

Consider two commuting random variables x and y

Positivity: $k(\cdot,\cdot)$ positive density

$$A = L^\infty(\mathbb{R}\times\mathbb{R}) = L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R}) = \{f(x,y)\}$$

$$\tau[f(x,y] = \int\int f(s,t)k(s,t)dsdt$$

$$\theta(f)(x,y) = \bar{f}(y,x)$$

$$A_+ = L^\infty(\mathbb{R})\otimes 1 = \{f(x)\}, \qquad A_- = 1\otimes L^\infty(\mathbb{R}) = \{f(y)\}$$

$$\tau[\theta(f)f] = \tau[\bar{f}(y)f(x)] = \int\int \bar{f}(s)f(t)k(s,t)dsdt \geq 0, \quad f = f(x)\in A_+$$

Meaning of reflection positivity

Consider two commuting random variables x and y

Positivity: $k(\cdot,\cdot)$ positive density

$$A = L^\infty(\mathbb{R}\times\mathbb{R}) = L^\infty(\mathbb{R})\otimes L^\infty(\mathbb{R}) = \{f(x,y)\}$$

$$\tau[f(x,y] = \int\int f(s,t)k(s,t)dsdt$$

Reflection positivity: $k(\cdot,\cdot)$ is positive kernel

$$\theta(f)(x,y) = \bar{f}(y,x)$$

$$A_+ = L^\infty(\mathbb{R})\otimes 1 = \{f(x)\}, \qquad A_- = 1\otimes L^\infty(\mathbb{R}) = \{f(y)\}$$

$$\tau[\theta(f)f] = \tau[\bar{f}(y)f(x)] = \int\int \bar{f}(s)f(t)k(s,t)dsdt \geq 0, \quad f = f(x)\in A_+$$

Meaning of reflection positivity

Consider two commuting random variables x and y.

Reflection positivity means then:

$$\mathbb{E}[f(x)\bar{f}(y)] \geq 0 \qquad \text{for all } f$$

Meaning of reflection positivity

Consider two commuting random variables x and y.

Reflection positivity means then:

$$\mathbb{E}[f(x)\bar{f}(y)] \geq 0 \qquad \text{for all } f$$

This is satisfied if x and y are i.i.d. But when else?

Meaning of reflection positivity

Consider two commuting random variables x and y.

Reflection positivity means then:

$$\mathbb{E}[f(x)\bar{f}(y)] \geq 0 \qquad \text{for all } f$$

This is satisfied if x and y are i.i.d. But when else?

Does this have any probabilistic meaning?

Bi-free probability

Consider pairs of faces: (x_i, y_i) for $i \in I$

$$A = \text{alg}(x_i, y_i; i \in I), \qquad A_i := \text{alg}(x_i, y_i)$$

[x_i and y_i don't need to commute; if they do, we call it bipartite]

Bi-free probability

Consider pairs of faces: (x_i, y_i) for $i \in I$

$$A = \text{alg}(x_i, y_i; i \in I), \qquad A_i := \text{alg}(x_i, y_i)$$

[x_i and y_i don't need to commute; if they do, we call it bipartite]

Bi-freeness = rule to calculate τ on A out of all $\tau|_{A_i}$

Bi-free probability

Consider pairs of faces: (x_i, y_i) for $i \in I$

$$A = \text{alg}(x_i, y_i; i \in I), \qquad A_i := \text{alg}(x_i, y_i)$$

[x_i and y_i don't need to commute; if they do, we call it bipartite]

Bi-freeness = rule to calculate τ on A out of all $\tau|_{A_i}$

- it is defined via realizing all x_i as left operators and all y_i as right operators acting on a free product Hilbert space

Bi-free probability

Consider pairs of faces: (x_i, y_i) for $i \in I$

$$A = \text{alg}(x_i, y_i; i \in I), \qquad A_i := \text{alg}(x_i, y_i)$$

[x_i and y_i don't need to commute; if they do, we call it bipartite]

Bi-freeness = rule to calculate τ on A out of all $\tau|_{A_i}$

- it is defined via realizing all x_i as left operators and all y_i as right operators acting on a free product Hilbert space
- in order to calculate τ for any product of the left and right variables do the following
  - move all left variables to the left, all right variables to the right
  - invert the order of the right variables
  - decompose now into the individual moments for each i according to the usual freeness rule
  - bring the left and right variables in each moment back into their original order

Bi-free probability

In particular one has for bi-free situation

- all left variables are free
- all right variables are free
- the left and the right variables corresponding to different indices are independent

Bi-free probability

In particular one has for bi-free situation

- all left variables are free
- all right variables are free
- the left and the right variables corresponding to different indices are independent

(x_1, y_1) and (x_2, y_2) bi-free means

- x_1, x_2 are free
- y_1, y_2 are free
- x_1 and y_2 are independent
- x_2 and y_1 are independent
- the relation between x_1 and y_1 can be arbitrarily prescribed
- the relation between x_2 and y_2 can be arbitrarily prescribed

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive
- this follows directly by definition, since everything is realized on Hilbert spaces

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive
- this follows directly by definition, since everything is realized on Hilbert spaces
- but: do we have a direct combinatorial proof for this???

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive
- this follows directly by definition, since everything is realized on Hilbert spaces
- but: do we have a direct combinatorial proof for this???

Reflection positivity for the left-right exchange $\theta(x_i) = y_i$

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive
- this follows directly by definition, since everything is realized on Hilbert spaces
- but: do we have a direct combinatorial proof for this???

Reflection positivity for the left-right exchange $\theta(x_i) = y_i$

- Question: If all $\theta|_{(x_i,y_i)}$ are reflection positive, is this then also true for θ on the bi-free product?

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive
- this follows directly by definition, since everything is realized on Hilbert spaces
- but: do we have a direct combinatorial proof for this???

Reflection positivity for the left-right exchange $\theta(x_i) = y_i$

- Question: If all $\theta|_{(x_i,y_i)}$ are reflection positive, is this then also true for θ on the bi-free product?
- Observation: This is true!

Bi-free probability: positivity questions

Positivity of the state

- if all $\tau|_{A_i}$ are positive, then τ on the bi-free product A is positive
- this follows directly by definition, since everything is realized on Hilbert spaces
- but: do we have a direct combinatorial proof for this???

Reflection positivity for the left-right exchange $\theta(x_i) = y_i$

- Question: If all $\theta|_{(x_i,y_i)}$ are reflection positive, is this then also true for θ on the bi-free product?
- Observation: This is true!
- Proof: This is just the fact that the Hadamard product of matrices preserves positive definiteness

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

$$\begin{aligned}
\to \varphi(x_1x_2x_2x_1y_2y_2) &= \varphi(x_1)\varphi(x_1)\varphi(x_2x_2y_2y_2) + \varphi(x_1x_1)\varphi(x_2x_2)\varphi(y_2y_2) \\
&\quad - \varphi(x_1)\varphi(x_2x_2)\varphi(x_1)\varphi(y_2y_2) \\
&\to \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2] \\
&\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2]
\end{aligned}$$

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

$$\begin{aligned}\to \varphi(x_1x_2x_2x_1y_2y_2) &= \varphi(x_1)\varphi(x_1)\varphi(x_2x_2y_2y_2) + \varphi(x_1x_1)\varphi(x_2x_2)\varphi(y_2y_2)\\ &\quad - \varphi(x_1)\varphi(x_2x_2)\varphi(x_1)\varphi(y_2y_2)\\ &\quad \to \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2]\\ &\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2]\end{aligned}$$

$$\begin{aligned}\tau[(x_2y_2x_1)^*(x_2y_2x_1)] &= \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2]\\ &\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2]\\ &= \tau[x_1]^2\tau[y_2x_2x_2y_2] + (\tau[x_1x_1] - \tau[x_1]^2)\tau[x_2x_2]\tau[y_2y_2]\end{aligned}$$

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

$$\begin{aligned}\tau[(x_2y_2x_1)^*(x_2y_2x_1)] &= \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2] \\ &\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2] \\ &= \tau[x_1]^2\tau[y_2x_2x_2y_2] + (\tau[x_1x_1] - \tau[x_1]^2)\tau[x_2x_2]\tau[y_2y_2]\end{aligned}$$

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

$$\begin{aligned}\tau[(x_2y_2x_1)^*(x_2y_2x_1)] &= \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2] \\ &\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2] \\ &= \tau[x_1]^2\tau[y_2x_2x_2y_2] + \big(\underbrace{\tau[x_1x_1] - \tau[x_1]^2}_{\geq 0}\big)\tau[x_2x_2]\tau[y_2y_2]\end{aligned}$$

Preservation of positivity

- if τ is positive on (x_1, y_1)

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

$$\begin{aligned}\tau[(x_2y_2x_1)^*(x_2y_2x_1)] &= \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2] \\ &\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2] \\ &= \tau[x_1]^2 \underbrace{\tau[y_2x_2x_2y_2]}_{\geq 0} + \big(\underbrace{\tau[x_1x_1] - \tau[x_1]^2}_{\geq 0}\big) \underbrace{\tau[x_2x_2]}_{\geq 0} \underbrace{\tau[y_2y_2]}_{\geq 0}\end{aligned}$$

Preservation of positivity

- if τ is positive on (x_1, y_1)
- and if τ is positive on (x_2, y_2)

Example of calculation of a positive moment

$$\tau[(x_2y_2x_1)^*(x_2y_2x_1)] = \tau[x_1y_2x_2x_2y_2x_1]$$

$$\begin{aligned}
\tau[(x_2y_2x_1)^*(x_2y_2x_1)] &= \tau[x_1]\tau[x_1]\tau[y_2x_2x_2y_2] + \tau[x_1x_1]\tau[x_2x_2]\tau[y_2y_2] \\
&\quad - \tau[x_1]\tau[x_2x_2]\tau[x_1]\tau[y_2y_2] \\
&= \tau[x_1]^2 \underbrace{\tau[y_2x_2x_2y_2]}_{\geq 0} + (\underbrace{\tau[x_1x_1] - \tau[x_1]^2}_{\geq 0}) \underbrace{\tau[x_2x_2]}_{\geq 0} \underbrace{\tau[y_2y_2]}_{\geq 0} \\
&\geq 0
\end{aligned}$$

Preservation of positivity

- if τ is positive on (x_1, y_1)
- and if τ is positive on (x_2, y_2)
- then the bi-free product is also positive

Example of calculation of reflection positive moment

$$\tau[\theta(x_1x_2)x_1x_2] = \tau[y_1y_2x_1x_2]$$

Example of calculation of reflection positive moment

$$\tau[\theta(x_1x_2)x_1x_2] = \tau[y_1y_2x_1x_2]$$

$$\to \varphi(x_1x_2y_2y_1) = \varphi(x_1y_1)\varphi(x_2y_2) \to \tau[y_1x_1]\tau[y_2x_2]$$

Example of calculation of reflection positive moment

$$\tau[\theta(x_1x_2)x_1x_2] = \tau[y_1y_2x_1x_2]$$

$$\to \varphi(x_1x_2y_2y_1) = \varphi(x_1y_1)\varphi(x_2y_2) \to \tau[y_1x_1]\tau[y_2x_2]$$

$$\tau[\theta(x_1x_2)x_1x_2] = \tau[y_1y_2x_1x_2] = \tau[y_1x_1]\tau[y_2x_2] = \tau[\theta(x_1)x_1] \cdot \tau[\theta(x_2)x_2]$$

Example of calculation of reflection positive moment

$$\tau[\theta(x_1 x_2) x_1 x_2] = \tau[y_1 y_2 x_1 x_2]$$

$$\tau[\theta(x_1 x_2) x_1 x_2] = \tau[\theta(x_1) x_1] \cdot \tau[\theta(x_2) x_2]$$

Example of calculation of reflection positive moment

$$\tau[\theta(x_1 x_2) x_1 x_2] = \tau[y_1 y_2 x_1 x_2]$$

$$\tau[\theta(x_1 x_2) x_1 x_2] = \underbrace{\tau[\theta(x_1) x_1]}_{\geq 0} \cdot \tau[\theta(x_2) x_2]$$

Preservation of reflection positivity

- if (x_1, y_1) is reflection positive

Example of calculation of reflection positive moment

$$\tau[\theta(x_1x_2)x_1x_2] = \tau[y_1y_2x_1x_2]$$

$$\tau[\theta(x_1x_2)x_1x_2] = \underbrace{\tau[\theta(x_1)x_1]}_{\geq 0} \cdot \underbrace{\tau[\theta(x_2)x_2]}_{\geq 0}$$

Preservation of reflection positivity

- if (x_1, y_1) is reflection positive
- and if (x_2, y_2) is reflection positive

Example of calculation of reflection positive moment

$$\tau[\theta(x_1x_2)x_1x_2] = \tau[y_1y_2x_1x_2]$$

$$\underbrace{\tau[\theta(x_1x_2)x_1x_2]}_{\geq 0} = \underbrace{\tau[\theta(x_1)x_1]}_{\geq 0} \cdot \underbrace{\tau[\theta(x_2)x_2]}_{\geq 0}$$

$$\geq 0 \quad \Longleftarrow \quad \geq 0 \quad \geq 0$$

Preservation of reflection positivity

- if (x_1, y_1) is reflection positive
- and if (x_2, y_2) is reflection positive
- then their bi-free product is also reflection positive

Summary

Summary

- reflection positivity is a nice and interesting property

Summary

- reflection positivity is a nice and interesting property
- bi-freeness seems to preserve this property

Summary

- reflection positivity is a nice and interesting property
- bi-freeness seems to preserve this property
- but it this good for anything???

Summary

- reflection positivity is a nice and interesting property
- bi-freeness seems to preserve this property
- but it this good for anything???

Thank you for your attention!