On operator valued R-diagonal and Haar unitary elements

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Haar unitary and R-diagonal elements

We work in a tracial von Neumann algebra $(\mathcal{M}, \tau)$. Namely, $\mathcal{M}$ is a von Neumann algebra and τ is a normal, faithful, tracial state on $\mathcal{M}$.

Note: some results described in this talk have non-tracial versions, but for simplicity we assume we are in the tracial setting.

A *Haar unitary element* is a unitary $u \in \mathcal{M}$ so that $\tau(u^n) = 0$ for all $n \in \mathbb{Z} \backslash \{0\}$. This entails that τ of spectral measure of u is Haar measure on the unit circle.

A *circular element* is $z \in \mathcal{M}$ where $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ are free, centered semicircular elements with the same second moment.

Theorem [Voiculsecu '90]

If z is a circular element, then it has polar decomposition $z = u|z|$, where u is a Haar unitary and u and $|z|$ are $*$-free from each other.

R-diagonality

Free cumulants of a family of elements in $\mathcal{M}$ were introduced by [Speicher '94].

[Nica, Speicher, '97] defined $a \in \mathcal{M}$ to be *R-diagonal* if all the cumulants of the pair (a, a^*) vanish except for those corresponding to alternating patterns $(a, a^*, \ldots, a, a^*)$ and $(a^*, a, \ldots, a^*, a)$ of even length.

Proposition [Nica, Shlyakhtenko, Speicher '01]

An element $a \in \mathcal{M}$ is R-diagonal if and only if a has the same $*$-distribution as uh (in some tracial von Neumann algebra), where u is a Haar unitary, $h \geq 0$ and where u and h are $*$-free.

In particular, Haar unitary elements and circular elements are R-diagonal.

R-diagonality (alternative formulation)

Definition [Boedihardjo, D., '18]

Given $\epsilon = (\epsilon(1), \ldots, \epsilon(n)) \in \{1, *\}^n$, the *maximal alternating interval partition* $\sigma(\epsilon)$ of ϵ is the partition into the largest possible interval blocks such that each block is alternating.

$$\text{E.g., } \epsilon = (\underbrace{*, 1, *}, \underbrace{*, 1}, \underbrace{1, *}) \implies \sigma(\epsilon) = \big\{\{1,2,3\},\{4,5\},\{6,7\}\big\}.$$

Prop. (equivalent, mild reformulation of part of [NiShlSp '01])

$a \in \mathcal{M}$ is R-diagonal if and only if

(a) all odd alternating moments vanish

(b) $\forall n \; \forall \epsilon \in \{1, *\}^n$,

$$\phi\left(\prod_{V \in \sigma(\epsilon)} \left(\left(\prod_{j \in V} a^{\epsilon(j)}\right) - \phi\left(\prod_{j \in V} a^{\epsilon(j)}\right)\right)\right) = 0.$$

Operator-valued noncommutative probability spaces

B-valued noncommutative probability spaces

Let (B, τ_B) be a tracial von Neumann algebra. We work in a tracial, B-valued W*-noncommutative probability space $(\mathcal{A}, \mathcal{E})$; this means $\mathcal{A}$ is a von Neumann algebra containing B as a unital subalgebra and $\mathcal{E} : \mathcal{A} \to B$ is a normal, faithful conditional expectation such that $\tau = \tau_B \circ \mathcal{E}$ is a trace on $\mathcal{A}$.

B-valued $*$-moments

The B-valued $*$-moments of $a \in \mathcal{A}$ are the multilinear maps $B \times \cdots \times B \to B$ of the form

$$(b_1, \ldots, b_{n-1}) \mapsto \mathcal{E}(a^{\epsilon(1)} b_1 a^{\epsilon(2)} b_2 \cdots a^{\epsilon(n-1)} b_{n-1} a^{\epsilon(n)})$$

for $n \in \mathbf{N}$ and $\epsilon = (\epsilon(1), \ldots, \epsilon(n)) \in \{1, *\}^n$.

Operator-valued R-diagonal elements [Śniady, Speicher '01]

B-valued free cumulants were defined by [Speicher '98].

B-valued R-diagonal elements were defined [Śniady, Speicher '01] in terms of B-valued cumulants.

Theorem [Śniady, Speicher '01]

An element $a \in \mathcal{A}$ is B-valued R-diagonal if and only if there exists an enlargement $(\widetilde{A}, \widetilde{\mathcal{E}})$ of $(A, \mathcal{E})$ and a unitary $u \in \widetilde{A}$ such that

- u commutes with B,
- $\{u, u^*\}$ is free from $\{a, a^*\}$ (over B),
- $\widetilde{\mathcal{E}}(u^k) = 0$ for all $k \in \mathbb{Z} \backslash \{0\}$,
- a and ua have the same B-valued $*$-moments.

Operator-valued R-diagonal elements (2)

Corollary

If a is a B-valued R-diagonal element with polar decomposition $a = v|a|$, then the partial isometry v is also B-valued R-diagonal.

Proof: The B-valued $*$-moments of v are determined by those of a. But with u as in the previous theorem, a and ua have the same B-valued $*$-monents. The polar decomopsition of ua is uv, so v and uv have the same B-valued $*$-moments.

Corollary

If a is a B-valued R-diagonal and if d is $*$-free from a (over B), then ad is B-valued R-diagonal.

Proof: Let u be a Haar unitary commuting with B and $*$-free from $\{a, d\}$. Then ua has the same $*$-moments as a, and ua is $*$-free from d, so uad has the same $*$-moments as ad.

Operator-valued R-diagonal elements (3)

Reformulation [Boedihardjo, D. '18]

An element $a \in A$ is R-diagonal if and only if

(a) all alternating moments of odd length vanish, for example those of the form

$$\mathcal{E}(ab_1a^*b_2ab_3a^*b_4a),$$

(b) $\forall n \; \forall \epsilon \in \{1,*\}^n$, $\forall b_1,\ldots,b_n \in B$,

$$\mathcal{E}\left(\prod_{V\in\sigma(\epsilon)}\left(\left(\prod_{j\in V}a^{\epsilon(j)}b_j\right)-\mathcal{E}\left(\prod_{j\in V}a^{\epsilon(j)}b_j\right)\right)\right)=0,$$

where $\sigma(\epsilon)$ is the maximal alternating interval partition associated to ϵ.

In particular, all B-valued $*$-moments of an R-diagonal element a are determined by the alternating B-valued $*$-moments.

Operator-valued R-diagonal elements (4)

Thus, the $*$-moments of an R-diagonal element a are determined by the $*$-moments having the even, alternating $*$-moments, denoted

$$\alpha_n(b_1, \ldots, b_{2n-1}) := \mathcal{E}(a^* b_1 a b_2 a^* b_3 a \cdots b_{2n-2} a^* b_{2n-1} a)$$

$$\beta_n(b_1, \ldots, b_{2n-1}) := \mathcal{E}(a b_1 a^* b_2 a b_3 a^* \cdots a^* b_{2n-2} a b_{2n-1} a^*).$$

B-valued circular elements ([Śniady '03] ?)

A B valued circular element is an R-diagonal element z whose B-valued cumulants vanish except for those of second order. In practice, these are the completely positive maps $B \to B$

$$\alpha_1(b) = \mathcal{E}(z^* b z), \quad \beta_1(b) = \mathcal{E}(z b z^*),$$

and then the higher even, alternating moments are determined recursively (via the moment-cumulant formula) for $n \geq 2$ by

$$\begin{aligned}
\alpha_n(b_1, \ldots, b_{n-1}) &= \mathcal{E}(a^* b_1 a b_2 a^* b_3 a \cdots b_{2n-2} a^* b_{2n-1} a) \\
&= \alpha_1(b_1) b_2 \alpha_{n-1}(b_3, \ldots, b_{2n-1}) \\
&\quad + \sum_{k=2}^{n-1} \alpha_1(b_1 \beta_{k-1}(b_2, \ldots, b_{2k-2}) b_{2k-1}) b_{2k} \\
&\qquad\qquad \alpha_{n-k}(b_{2k+1}, \ldots, b_{2n-1}) \\
&\quad + \alpha_1(b_1 \beta_{n-1}(b_2, \ldots, b_{2n-1}) b_{2n-1})
\end{aligned}$$

and likewise, reversing the roles of β and α.

B-valued circular elements (2)

Given any two completely positive maps α_1 and β_1 from B to B, there exists a unique corresponding B-valued circular element z such that

$$\alpha_1(b) = \mathcal{E}(z^* b z), \quad \beta_1(b) = \mathcal{E}(z b z^*).$$

Proposition [Boedihardjo, D. '18]

The B-valued circular element z can be realized in a tracial B-valued W*-noncommutative probability space if and only if for a faithful tracial state τ_B on B, we have

$$\tau_B(\alpha_1(b_1)b_2) = \tau_B(b_1\beta_1(b_2))$$

for all $b_1, b_2 \in B$.

Example [Boedihardjo, D. '18]

Take $B = \mathbb{C}^2$ endowed with the equal weight trace and consider the completely positive maps $B \to B$

$$\alpha_1(\lambda_1, \lambda_2) = \left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2} + \lambda_2\right)$$
$$\beta_1(\lambda_1, \lambda_2) = \left(\frac{\lambda_1 + \lambda_2}{2}, \lambda_2\right).$$

Let z be the corresponding (tracial) circular element. We compute the distribution of z^*z with respect to $\tau_B \circ \mathcal{E}$ and see that it has zero kernel, so it has polar decomposition $z = u|z|$, with u unitary.

We cannot have that u and $|z|$ are $*$–free over B, because $\beta_1(1) = 1$ while $\alpha_1(1) \neq 1$.

Conclusion

In the B-valued setting, circular elements and (more generally) R-diagonal elements need not have free polar decompositions.

Classes of B-valued Haar unitaries

We work in $(\mathcal{A}, \mathcal{E})$ as before.

Definition

Let $u \in \mathcal{A}$ be a unitary element. We say u is

(a) a *Haar unitary element* if $\mathcal{E}(u^n) = 0$ for all $n \in \mathbb{Z} \backslash \{0\}$,

(b) a *balanced unitary element* if $\mathcal{E}(u^{\epsilon(1)} b_1 u^{\epsilon(2)} b_2 \cdots u^{\epsilon(n-1)} b_{n-1} u^{\epsilon(n)}) = 0$ whenever $\#\{j \mid \epsilon(j) = *\} \neq \#\{j \mid \epsilon(j) = 1\}$ and $b_1, \ldots, b_{n-1} \in B$,

(c) an *R-diagonal unitary element* if u is also R-diagonal,

(d) a *normalizing Haar unitary element* if u is Haar unitary and if, for some automorphism θ of B and all b in B, $u^* b u = \theta(b)$.

Theorem

(d) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a), and none of the reverse implications hold.

Example: a Haar unitary that is not balanced

Let τ be the trace on $C(\mathbb{T})$ given by integration with respect to Haar measure on the unit circle $\mathbb{T}$. Let $v \in C(\mathbb{T})$ be the identity map on $\mathbb{T}$ (thus, a Haar unitary with respect to τ). Let $\mathcal{A} = M_2(C(\mathbb{T})) \cong M_2(\mathbb{C}) \otimes C(\mathbb{T})$ and let $B \subseteq \mathcal{A}$ be the diagonal matrices having scalar entries, so $B \cong \mathbb{C}^2$. Let $\mathcal{E} : \mathcal{A} \to B$ be

$$\mathcal{E}\left(\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\right) = \begin{pmatrix} \tau(f_{11}) & 0 \\ 0 & \tau(f_{22}) \end{pmatrix}.$$

Let $p = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and let $u = p \otimes v + (1-p) \otimes v^*$. Then u is Haar unitary with respect to $\mathcal{E}$, but $\mathcal{E}(ue_{11}u) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so u is not a balanced unitary.

Example: a balanced unitary that is not R-diagonal

Let τ be the canonical trace on $C^*(\mathbb{Z}\times\mathbb{Z})$, with $v, w \in C^*(\mathbb{Z}\times\mathbb{Z})$ commuting Haar unitaries. Let $\mathcal{A} = M_2(C^*(\mathbb{Z}\times\mathbb{Z})) \cong M_2(\mathbb{C}) \otimes C^*(\mathbb{Z}\times\mathbb{Z})$ and let $B \subseteq \mathcal{A}$ be the diagonal matrices having scalar entries, so $B \cong \mathbb{C}^2$. As before, let $\mathcal{E} : \mathcal{A} \to B$ be

$$\mathcal{E}\left(\left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \tau(f_{11}) & 0 \\ 0 & \tau(f_{22}) \end{smallmatrix}\right).$$

and $p = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Let $u = p \otimes v + (1-p) \otimes w$. Then it is straightforward to compute that u is a balanced unitary, but when $b_1 = b_2 = b_3 = 1 \oplus 0 \in B$ (identified with the matrix unit $e_{11} \in M_2(\mathbb{C})$), we find

$$E((u^* b_1 u - E(u^* b_1 u))\, b_2\, (u b_3 u^* - E(u b_3 u^*))) = \frac{1}{8}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \neq 0.$$

Thus, u is not an R-diagonal element.

Example: an R-diagonal unitary that is not normalizing

Let $B = \mathbb{C}^2$ and let z be the B-valued circular element corresponding to the maps

$$\alpha_1(\lambda_1, \lambda_2) = \left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2} + \lambda_2 \right)$$
$$\beta_1(\lambda_1, \lambda_2) = \left(\frac{\lambda_1 + \lambda_2}{2}, \lambda_2 \right).$$

We know that z has polar decomposition $z = u|z|$, with u unitary. By the Corollary to [Śniady, Speicher '01], this u is an R-diagonal unitary element.

Example: an R-diagonal unitary that is not normalizing (2)

However, if u were normalizing with $u^*bu = \theta(b)$, then we would have, for every $x \in \mathcal{A}$, $\mathcal{E}(x) = 0 \implies \mathcal{E}(uxu^*) = 0$, since

$$\tau_B(\mathcal{E}(uxu^*)^*\mathcal{E}(uxu^*)) = \tau_B \circ \mathcal{E}(ux^*u^*\mathcal{E}(uxu^*)) = \tau_B \circ \mathcal{E}(x^*u^*\mathcal{E}(uxu^*)u) \\ = \tau_B \circ \mathcal{E}(x^*\theta(\mathcal{E}(uxu^*))) = \tau_B(\mathcal{E}(x^*)\theta(\mathcal{E}(uxu^*))) = 0.$$

This implies, for all $x \in \mathcal{A}$, $\mathcal{E}(uxu^*) = \theta^{-1}(\mathcal{E}(x))$.

Now we get

$$\mathcal{E}(zz^*bzz^*) = \mathcal{E}(u|z|^2u^*bu|z|^2u^*) = \theta^{-1}(\mathcal{E}(|z|^2\theta(b)|z|^2)) \\ = \theta^{-1}(\mathcal{E}(z^*z\theta(b)z^*z)).$$

However, $\mathcal{E}(zz^*bzz^*)$ and $\mathcal{E}(z^*z\theta(b)z^*z)$ can be computed in terms of the defining completely positive maps α_1 and β_1, and we easily see that the above equality fails to hold (for both possible automorphisms θ) when $b = (1,0) \in B$.

Motivating Question: what can B-valued R-diagonal unitaries look like if they are not normalizing?

One example: we understand the $\mathbb{C}^2$-valued circular element z of the previous example quite well and we know $z = u|z|$, where u is an R-diagonal unitary that is not normalizing (and also not free from $|z|$).

We know the distribution of $|z|^2$ with respect to $\tau_B \circ \mathcal{E}$. Can we use this to find the $\mathbb{C}^2$-valued distribution of $|z|$ and thereby to describe the $\mathbb{C}^2$-valued distribution of u?

Another idea: suppose there exists a B-valued circular element z, and suppose $z = u|z|$ with u and $|z|$ $*$-free over B. Let us call this a *free polar decomposition*. Perhaps freeness would help us to find out more about the $*$-moments of u from those of z. Of course, if u is already normalizing (of B), then we are not so interested in this case.

Bipolar decompositions

Unfortunately, we don't understand well, in terms of cumulants, conditions for a polar decomposition of a B-valued R-diagonal to have a free and normalizing unitary part. (There is a theorem in [Boedihardjo, Dykema '18] that purports to do so, but it is erroneous. See [erratum '23].) Instead, we turn to bipolar decompositions.

Definition

Let $(\mathcal{A}, \mathcal{E})$ be a B-valued W*-noncommutative probability space and let $a \in \mathcal{A}$. A *bipolar decomposition* of a is a pair (u, x) of elements in some B-valued W*-noncommutative probability space $(\mathcal{A}', \mathcal{E}')$, such that u is a partial isometry, x is self-adjoint and ux has the same $*$-moments as a.

Bipolar decompositions are not unique. Examples include polar decompositions. "Bipolar" refers to the positive and negative directions of $\mathbb{R}$. If (u, x) is a bipolar decomposition, then $x = s|x|$ for a symmetry (namely, a self-adjoint unitary) s that commutes with x. Thus, $(us)|x|$ is a polar decomposition.

Bipolar decompositions (2)

Definition

A bipolar decomposition (u, x) in $(\mathcal{A}', \mathcal{E}')$ of an element a is

- *minimal* if u^*u equals the support projection of x;
- *unitary* if u is a unitary element;
- *tracial* if there is a normal tracial state τ_B on B so that $\tau_B \circ \mathcal{E}'$ is a trace on the $*$-algebra generated by u and x;
- *standard* if there is a symmetry $s \in \mathcal{A}'$ such that $x = s|x|$ and such that s commutes with x, with u and with every $b \in B$;
- *even* if all odd moments of x vanish, namely, if $\mathcal{E}'(xb_1xb_2 \cdots xb_{2n}x) = 0$ for all $n \geq 1$ and $b_1, \ldots, b_{2n} \in B$.
- *free* if u and x are $*$-free over B (with respect to the conditional expectation $\mathcal{E}'$);
- *normalizing* if it is unitary and u normalizes the algebra B, namely, if $u^*bu = \theta(b)$ for every $b \in B$, for some automorphism θ of B.

Bipolar decompositions (3)

Lemma

Every B-valued element, a, has a bipolar decomposition that is standard, even and minimal. If a is tracial, then this bipolar decomposition can also be taken to be tracial.

Proof: If $a = v|a|$ is the polar decomposition, then take $u = v \oplus (-v)$ and $x = |a| \oplus (-|a|)$.

Lemma on R-diagonal unitaries in bipolar decompositions

Suppose a B-valued R-diagonal element a has a bipolar decomposition (v, x). Then a also has a bipolar decomposition (v', x'), where x' has the same distribution as x and where v' is B-valued R-diagonal. Furthermore, if (v, x) is tracial, unitary, minimal, standard, free or normalizing, then also (v', x') can be taken to be tracial, unitary, minimal, standard, free, or normalizing, respectively.

Free, normalizing bipolar decompositions of R-diagonals

Recall our notation for the even alternating moments:

$$\alpha_n(b_1,\ldots,b_{2n-1}) := \mathcal{E}(a^*b_1ab_2a^*b_3a\cdots b_{2n-2}a^*b_{2n-1}a)$$

$$\beta_n(b_1,\ldots,b_{2n-1}) := \mathcal{E}(ab_1a^*b_2ab_3a^*\cdots a^*b_{2n-2}ab_{2n-1}a^*).$$

Theorem [Boedihardjo, D. '18] (but using current terminology)

Let a be a B-valued R-diagonal element. Then a has a free, normalizing bipolar decomposition (u,x) with corresponding automorpihsm $u^*bu = \theta(b)$ if and only if

$$\alpha_n(b_1,\theta(b_2),b_3,\ldots,\theta(b_{2n-2}),b_{2n-1}) = \theta\big(\beta_n(\theta(b_1),b_2,\theta(b_3),\ldots,b_{2n-2},\theta(b_{2n-1}))\big)$$

for all n and $b_1,\ldots,b_{2n-1}\in B$.

If a is actually B-valued circular, then the above condition beomes $\alpha_1(b) = \theta(\beta_1(\theta(b)))$ for all $b\in B$.

Example with a free normalizing bipolar decomposition but no normalizing polar decomposition

Let z be a copy of Voiculescu's circular element ($\mathbb{C}$-valued) in a W*-noncommutative probability space $(\mathcal{A}_0, \tau_0)$. Suppose (B, τ_B) is a tracial von Neumann algebra, $B \neq \mathbb{C}$. Let $(\mathcal{A}, \tau) = (\mathcal{A}_0, \tau_0) * (B, \tau_B)$ be the free product of von Neumann algebras and let $\mathcal{E} : \mathcal{A} \to B$ be the τ-preserving conditional expectation onto B. By [Śniady, Speicher '01], a is also B-valued circular in $(\mathcal{A}, \mathcal{E})$, with corresponding completely positive maps $\alpha_1(b) = \beta_1(b) = \tau_B(b)1$. By the previous Theorem, for every τ_B-preserving automorphism θ of B, there is a free, normalizing bipolar decomposition (u, x) of a with $u^* b u = \theta(b)$ for all $b \in B$. However, the polar decomposition of a is $a = v|a|$ with $v \in \mathcal{A}_0$ that is Haar unitary with respect to τ_0. Thus, v is free from B and cannot normalize B.

More on that example

The previous example can be concretely realized in the free product (over $\mathbb{C}$)

$$(\mathcal{A}, \tau) = (B \rtimes_\theta \mathbb{Z}, \tau_B \circ E) * (L^\infty[-2, 2], \tau_2)$$

where $E : B \rtimes_\theta \mathbb{Z} \to B$ is the conditional expectation and τ_2 is by integration against Lebesgue measure. Let $\mathcal{E} : \mathcal{A} \to B$ be the τ-preserving conditional expectation.

Now take a semicircular element $x \in L^\infty[-2, 2]$ and a symmetry s so that $x = s|x|$. Let $u \in B \rtimes_\theta \mathbb{Z}$ be the Haar unitary implimenting θ. Then $z = us|x|$ is a circular element with respect to τ, is $*$-free from B, us and $|x|$ are $*$–free from each other and (u, x) is a free, normalizing bipolar decomposition for z.

But under a nondegeneracy condition:

Theorem

Let a be a B-valued random variable in a B-valued W*-noncommutative probability space $(\mathcal{A}, \mathcal{E})$ and assume that either of the subspaces

$$\mathrm{span}\,\{\mathcal{E}((a^*a)^k) \mid k \geq 0\} \text{ or } \mathrm{span}\,\{\mathcal{E}((aa^*)^k) \mid k \geq 0\}$$

is weakly dense in B. Suppose that a has free bipolar decompositions (u, x) and $(\tilde{u}, \tilde{x})$ in B-valued W*-noncommutative probability spaces (A', E') and $(\widetilde{A}, \widetilde{E})$, respectively, with $\widetilde{E}$ faithful. Suppose u and $\tilde{u}$ are unitaries satisfying $E'(u) = 0 = \widetilde{E}(\tilde{u})$ and suppose that u normalizes B. Then $\tilde{u}$ normalizes B, and induces the same automorphism, namely, $\tilde{u}^* b \tilde{u} = u^* b u$ for all $b \in B$.

Motivating question and answer in a special case

Question

Suppose a tracial B-valued circular element a has a tracial, free, bipolar decomposition (u, x) with u unitary. Must a also have a free bipolar decomposition that is normalizing?

Note that under the nondegeneracy hypothesis, that

$$\mathrm{span}\,\{\mathcal{E}((a^*a)^k) \mid k \geq 0\} \text{ or } \mathrm{span}\,\{\mathcal{E}((aa^*)^k) \mid k \geq 0\}$$

is weakly dense in B, we would conclude that every free bipolar decomposition of a is normalizing.

Theorem

Yes, when $B = \mathbb{C}^2$.

The theorem covering the case $B = \mathbb{C}^2$

In particular, we prove that if a is tracial $\mathbb{C}^2$-valued circular in $(\mathcal{A}, \mathcal{E})$ with corresponding completely positve maps α_1 and β_1 and if a has a free bipolar decomposition (u, x) with u unitary, then $\alpha_1 = \theta \circ \beta_1 \circ \theta$ for one of the two automorphisms θ of $\mathbb{C}^2$.

Method of proof: arduous calculation.

Just to give a taste of this: the defining maps α_1 and β_1, as well as the trace τ_B, are defined in terms of certain parameters. Taking $a = ux$, we use the freeness assumption to obtain certain relations, e.g.,

$$\mathcal{E}((aa^*)^n) = \mathcal{E}(u\mathcal{E}(x^{2n})u^*) = \mathcal{E}(u\mathcal{E}((a^*a)^n)u^*)$$

and more complicated ones, e.g., involving $\mathcal{E}((aa^*)^n(a^*a)^m(aa^*)^k)$. We can dispense with a degnerate case and assume without loss of generality $\mathrm{span}\,\{1_B, \mathcal{E}(a^*a)\} = B$. Using all of this and more, we obtain some nasty-looking algebraic relations among the aforementioned parameters. With help of Mathematica, we are able to show that we must have $\alpha_1 = \theta \circ \beta_1 \circ \theta$ for one of the θ.

Open questions:

Assume that

$$\mathrm{span}\,\{\mathcal{E}((a^*a)^k) \mid k \geq 0\} \text{ or } \mathrm{span}\,\{\mathcal{E}((aa^*)^k) \mid k \geq 0\}$$

is weakly dense in B,

Question

Suppose a tracial B-valued circular element a has a tracial, free, bipolar decomposition (u, x) with u unitary. Must u normalize B?

Question

Suppose a tracial B-valued R-diagonal element a has a tracial, free, bipolar decomposition (u, x). Must its polar decomposition be free?

Thanks for your attention! Selected References (chronological):

[1] D. Voiculescu, *Circular and semicircular systems and free product factors*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, 1990, pp. 45–60.

[2] R. Speicher, *Multiplicative functions on the lattice of noncrossing partitions and free convolution*, Math. Ann. **298** (1994), 611–628.

[3] A. Nica and R. Speicher, *R-diagonal pairs—a common approach to Haar unitaries and circular elements*, Free Probability Theory (Waterloo, Ontario, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., 1997, pp. 149–188.

[4] R. Speicher, *Combinatorial Theory of the Free Product with Amalgamation and Operator–Valued Free Probability Theory*, Mem. Amer. Math. Soc., vol. 627, 1998.

[5] A. Nica, D. Shlyakhtenko, and R. Speicher, *R-diagonal elements and freeness with amalgamation*, Canad. J. Math. **53** (2001), 355–381.

[6] P. Śniady and R. Speicher, *Continuous family of invariant subspaces for R-diagonal operators*, Invent. Math. **146** (2001), 329–363.

[7] P. Śniady, *Multinomial identities arising from free probability theory*, J. Combin. Theory Ser. A **101** (2003), 1–19.

[8] M. Boedihardjo and K. Dykema, *On algebra-valued R-diagonal elements.*, Houston J. Math. **44** (2018), 209–252;: *Erratum* **49** (2023), 157–158.

[9] K. Dykema and J. A. Griffin, *On operator valued Haar unitaries and bipolar decompositions of R-diagonal elements*, Integral Equations Operator Theory, to appear, available at `arXiv:2306.17333(w/Mathematica)`.