Nuclear C^*-algebras and generalized inductive limits

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based in part on joint work with W. Winter
and with N. Galke, L. van Luijk, and A. Stottmeister

UC Berkeley Probabilistic Operator Algebra Seminar

Part I: Inductive limits and AF algebras

Inductive limits of C^*-algebras

An inductive system of C^*-algebras consists of a sequence $(A_n)_n$ of C^*-algebras together with connecting *-homomorphisms

$$\begin{tikzcd} A_0 \arrow[r, "\rho_{1,0}"] \arrow[rr, bend left, "\rho_{2,0}"] & A_1 \arrow[r, "\rho_{2,1}"] & A_2 \arrow[r] & \ldots \end{tikzcd}$$

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The quotient map $\prod_n A_n \to {}^{\prod A_n}/_{\bigoplus A_n}$ induces *-homomorphisms $\rho_k : A_k \to {}^{\prod A_n}/_{\bigoplus A_n}$ for each $k \geq 0$ by

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The inductive limit of the system $(A_n, \rho_{m,n})$ is the C^*-algebra

$$\varinjlim(A_n, \rho_{m,n}) := \overline{\bigcup_k \rho_k(A_k)} \subset \prod A_n / \bigoplus A_n.$$

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The CAR algebra:

$$\mathrm{M}_2 \xhookrightarrow{a \mapsto a \oplus a} \mathrm{M}_4 \hookrightarrow \ldots \hookrightarrow \overline{\bigcup_k \mathrm{M}_{2^k}} =: \mathrm{M}_{2^\infty}$$

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Alternatively, an AF C^*-algebra is one that contains an ascending sequence of finite-dimensional subalgebras with norm-dense union, which makes its von Neumann analogue a little more apparent.

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Definition

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Example

The hyperfinite II_1-factor $\mathcal{R}$.

$$\mathrm{M}_2 \xhookrightarrow{a \mapsto a \oplus a} \mathrm{M}_4 \hookrightarrow \ldots \hookrightarrow \overline{\bigcup_k \mathrm{M}_{2^k}}^{wk^*} = \mathcal{R}$$

Semi-discrete von Neumann Algebras

Definition

A separably acting von Neumann algebra $\mathcal{M}$ is semi-discrete iff there exists a sequence of finite-dimensional von Neumann algebras $(F_n)_{n\in\mathbb{N}}$ and unital completely positive (ucp) maps $\mathcal{M} \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} \mathcal{M}$ such that $\varphi_n \circ \psi_n \to \mathrm{id}_{\mathcal{M}}$ pointwise wk*.

This gives a sequence of (wk*-)approximately commuting diagrams.

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Example

Any AFD von Neumann algebra is semi-discrete.

Nuclear C^*-algebras

Theorem/Definition (Choi–Effros '78; Kirchberg '77)

A separable C^*-algebra A is nuclear iff there exists a sequence of finite-dimensional C^*-algebras $(F_n)_{n\in\mathbb{N}}$ and completely positive contractive (cpc) maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ such that $\varphi_n \circ \psi_n \to \mathrm{id}_A$ pointwise in norm.

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- $C(X)$ where $\dim(X) \geq 1$.
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Even though a direct analogue to Connes' result, i.e., "nuclear $\Rightarrow$ AF", is out of the question, any system of cpc approximations of a nuclear C^*-algebra gives rise to something very much like an inductive system.

From CPAP to an “inductive system”

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Forming the limit

Forming the limit with *-homomorphisms

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Suppose $A = \overline{\bigcup F_n}$ with $\varphi_n : F_n \hookrightarrow A$ the inclusion and $\psi_n : A \to F_n$ a conditional expectation. Then the $\rho_{n+1,n}$ are *-homomorphisms,

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& & & & & & \overline{\bigcup \rho_n(F_n)} \quad \subseteq \quad \frac{\prod F_j}{\bigoplus F_j}
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$$

and $(F_n, \rho_{n+1,n})_n$ is an inductive system with limit

$$\varinjlim(F_n, \rho_{n+1,n}) := \overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}.$$

Back to A

The $(\psi_n)_n$ induce a cpc map $\Psi : A \to \prod F_j / \bigoplus F_j$.

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$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \xrightarrow[n\to\infty]{} 0, \ \forall\ a, b \in A$$

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& F_0 \arrow[ur, "\varphi_0"] \arrow[rr, "\rho_{1,0}"] \arrow[drrrrrr, bend right, "\rho_0"] & & F_1 \arrow[ur, "\varphi_1"] \arrow[rr, "\rho_{2,1}"] \arrow[drrrr, bend right, "\rho_1"] & & F_2 \arrow[ur, "\varphi_2"] \arrow[r] \arrow[drr, bend right, "\rho_2"] & \dots & & & & \\
& & & & & & & \overline{\bigcup \rho_n(F_n)} & \subseteq & \frac{\prod F_j}{\bigoplus F_j} &
\end{tikzcd}
$$

Ψ is isometric	since	$(\psi_n)_n$ are approx isometric
$\Psi(A) = \overline{\bigcup \rho_n(F_n)}$	since	$\rho_{m,n} = \psi_m \circ \varphi_n, \ \forall\ m > n$
Ψ is a *-homomorphism	since	$(\psi_n)_n$ are approx mult

$\rightsquigarrow$ $\Psi : A \to \overline{\bigcup \rho_n(F_n)}$ is a $*$-isomorphism.

Forming the limit with cpc maps

When the $\rho_{n+1,n}$ are cpc maps,

$$
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A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"] & & A \arrow[rr] \arrow[dr, "\psi_2"] & & \dots \\
& F_0 \arrow[ur, "\varphi_0"] \arrow[rr, "\rho_{1,0}", "cpc"'] & & F_1 \arrow[ur, "\varphi_1"] \arrow[rr, "\rho_{2,1}", "cpc"'] & & F_2 \arrow[ur, "\varphi_2"] \arrow[r] & \dots
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Forming the limit with cpc maps

When the $\rho_{n+1,n}$ are cpc maps, they still induce cpc maps $\rho_n : F_n \to \prod F_j / \bigoplus F_j$ with $\rho_n(x) = [(\rho_{m,n}(x))_{m>n}]$.

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\\
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& & & & & & \overline{\bigcup \rho_n(F_n)} & \subseteq & \frac{\prod F_j}{\bigoplus F_j}
\end{tikzcd}
$$

We can still form the limit of the system $(F_n, \rho_{n+1,n})_n$

$$\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j},$$

which is now just a closed self-adjoint subspace.

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How does it relate to A?

Back to A

The $(\psi_n)_n$ induce a cpc map $\Psi : A \to \prod F_j / \bigoplus F_j$.

$$
\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"'] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"'] & & A \arrow[rr] \arrow[dr, "\psi_2"'] & & \cdots \arrow[r] & A \arrow[dd, "\Psi"] \\
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& & & & & & \overline{\bigcup \rho_n(F_n)} \arrow[r, phantom, "\subseteq"] & \dfrac{\prod F_j}{\bigoplus F_j}
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Ψ is completely isometric since $(\psi_n^{(r)})_n$ are approx isometric $\forall\ r \geq 1$

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$$\|\rho_{m,n} - \psi_m \circ \varphi_n\| \xrightarrow[m,n\to\infty]{} 0$$

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Any system of cpc approximations admits a summable subsystem, so we assume our system is summable.

Back to A

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Ψ is a *-homomorphism	if	$(\psi_n)_n$ are approx mult

This can only happen if A is quasidiagonal (QD).

Back to A

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\begin{tikzcd}
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$\rightsquigarrow$ $\Psi : A \to \overline{\bigcup \rho_n(F_n)}$ is a complete order isomorphism (coi).

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That means Ψ is completely isometric and cp with cp inverse.

Back to A

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$$\begin{tikzcd}
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& F_0 \arrow[ur, "\varphi_0"] \arrow[rr, "\rho_{1,0}"] \arrow[ddrrrrrr, phantom] \arrow[drrrrrr, bend right, "\rho_0"'] & & F_1 \arrow[ur, "\varphi_1"] \arrow[rr, "\rho_{2,1}"] \arrow[drrrr, bend right, "\rho_1"'] & & F_2 \arrow[ur, "\varphi_2"] \arrow[r] \arrow[drr, bend right, "\rho_2"'] & \ldots & & & \\
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& & & & & & & \overline{\bigcup \rho_n(F_n)} \arrow[uur, leftrightarrow, "\Psi", "\mathrm{coi}"'] & \subseteq & \frac{\prod F_j}{\bigoplus F_j}
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Since any coi between C^*-algebras is automatically a *-isomorphism, the coi class of a C^*-algebra captures its *-isomorphism class.

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Since any coi between C^*-algebras is automatically a *-isomorphism, the coi class of a C^*-algebra captures its *-isomorphism class.

Moreover, by equipping $\overline{\bigcup \rho_n(F_n)}$ with the product

$$\Psi(a) \bullet \Psi(b) := \Psi(ab), \ \forall\ a, b \in A,$$

we get a C^*-algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$, which is *-isomorphic to A.

Part II: cpc systems and nuclearity

cpc systems

Somehow the system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ produced, not a C^*-algebra, but a space completely order isomorphic to a nuclear C^*-algebra.

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Definition

We call a sequence of C^*-algebras $(A_n)_n$ together with cpc connecting maps $\rho_{n+1,n} : A_n \to A_{n+1}$ a cpc system, denoted $(A_n, \rho_{n+1,n})_n$. When the A_n are all finite-dimensional, we call the system finite-dimensional.

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In one sense this is a special case of Blackadar and Kirchberg's Generalized Inductive Systems. In another sense, it is a generalization.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

Nuclearity

Proposition (C.–Winter, C.)

If the limit of a finite-dimensional cpc system is coi to a C^**-algebra* A*, then* A *is nuclear.*

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This follows readily from Ozawa and Sato's One-Way-CPAP, which allows one to determine whether a given C^*-algebra A is nuclear by finding a certain family of cpc maps $\{\varphi_\lambda : F_\lambda \to A\}_\lambda$ from finite-dimensional C^*-algebras.

One Way CPAP

Theorem (Ozawa '02, Sato '21)

A C^**-algebra* A *is nuclear iff there exists a net* $(\varphi_\lambda : F_\lambda \to A)_{\lambda\in\Lambda}$ *of cpc maps from finite-dimensional* C^**-algebras such that the induced cpc map*

$$\begin{tikzcd} \prod_\lambda F_\lambda \arrow[r, "(\varphi_\lambda)_\lambda"] \arrow[d, two heads] & \ell^\infty(\Lambda, A) \arrow[d, two heads] \\ \prod_\lambda F_\lambda / \bigoplus_\lambda F_\lambda \arrow[r, "\Phi"] & \ell^\infty(\Lambda, A)/c_0(\Lambda, A) \end{tikzcd}$$

satisfies $A^1 \subset \Phi\left(\left(\frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda}\right)^1\right)$.

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To get the φ_n in our case:

$$\begin{tikzcd} F_0 \arrow[r, "\rho_{1,0}"] \arrow[drrr, bend right, "\rho_0"'] & F_1 \arrow[r, "\rho_{2,1}"] \arrow[drr, bend right, "\rho_1"'] & F_2 \arrow[r] \arrow[dr, bend right, "\rho_2"] & \dots \\ & & & \overline{\bigcup \rho_n(F_n)} \arrow[d, "\Psi"', "\text{coi}"] \quad \subseteq \quad \frac{\prod F_j}{\bigoplus F_j} \\ & & & A \end{tikzcd}$$

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NF systems (Blackadar and Kirchberg)

Definition (Blackadar–Kirchberg '97)

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$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

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Think of this as saying that for $m > n > M$, the maps $\rho_{m,n}$ become more multiplicative on $\rho_{n,k}(x)$ and $\rho_{n,k}(y)$.

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,n}\big(\rho_{n,k}(x)\big)\rho_{m,n}\big(\rho_{n,k}(y)\big)\| < \varepsilon.$$

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Moreover, for any nuclear and QD C^*-algebra A, there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately multiplicative so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is NF and its limit is *-isom to A.

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"] & & A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_2"] & & \dots \\
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\end{tikzcd}$$

Remark

For any summable system of cpc approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$, $(\psi_n)_n$ are approximately multiplicative iff $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is NF.

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One natural step down comes from order cp zero maps.

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Proposition (Winter–Zacharias '09)

Let A and B be C^*-algebras with A unital. A cp map $\varphi : A \to B$ is order zero iff $\varphi(a)\varphi(b) = \varphi(1_A)\varphi(ab)$ for all $a, b \in A$.

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Note that a unital cp order zero map is automatically a *-homomorphism.

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A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is NF if it is asymptotically multiplicative, meaning that for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n > M$

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ of an NF system is a C^*-subalgebra with

$$\rho_k(x)\rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \ \forall\ k \geq 0, x, y \in F_k.$$

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ of a CPC^* system is completely order isomorphic to the C^*-algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$ with

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Side-by-side

Definition (Blackadar–Kirchberg '97)

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The following are equivalent for a separable C^*-algebra A:

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Moreover, for any nuclear and QD C^*-algebra A, there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately multiplicative so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is NF and its limit is *-isom to A.

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Hierarchy

Example (C.–Winter)

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Example

Any separable nuclear non-QD C^*-algebra admits a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is CPC^* and not NF.

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Remark

If the connecting maps $\rho_{n+1,n} : F_n \to F_{n+1}$ are unital, then a CPC^* system is automatically NF.

Part III: C^*-encoding systems

Back to our motivating observations

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Though systems $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations with $(\psi_n)_n$ approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC^*-systems.

Systems from Følner sequences

For a countable, discrete, amenable group Γ, we can use any Følner sequence $(\mathcal{G}_n)_n$ to construct a system of ucp approximations of $\mathrm{C}^*_\lambda(\Gamma)$:

$$\begin{tikzcd}
\mathrm{C}^*_\lambda(\Gamma) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"] & & \mathrm{C}^*_\lambda(\Gamma) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"] & & \mathrm{C}^*_\lambda(\Gamma) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_2"] & & \dots \\
& M_{\mathcal{G}_0} \arrow[ur, "\varphi_0"] & & M_{\mathcal{G}_1} \arrow[ur, "\varphi_1"] & & M_{\mathcal{G}_2} \arrow[ur, "\varphi_2"] & \dots
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\end{tikzcd}$$

Identifying $M_{\mathcal{G}_n} \cong P_n B(\ell^2(\Gamma)) P_n$ with $P_n = \mathrm{proj}_{\mathrm{span}\{\delta_g | g \in \mathcal{G}_n\}}$, we set

$$\psi_n(x) = P_n x P_n \text{ for } x \in \mathrm{C}^*_\lambda(\Gamma) \subset B(\ell^2(\Gamma))$$

and

$$\varphi_n(e_{g,h}) = \tfrac{1}{|F_n|}\lambda_{gh^{-1}}$$

where $\{e_{g,h} \mid g, h \in \mathcal{G}_n\} \subset M_{\mathcal{G}_n}$ are the matrix units.

Example $\mathrm{C}^*_\lambda(\mathbb{Z})$

For $\Gamma = \mathbb{Z}$ and Følner sets $\mathcal{G}_n = \{0, ..., n-1\}$, we have $M_{\mathcal{G}_n} = M_n$ with matrix units $\{e_{i,j}\}_{i,j=0}^{n-1}$. Then for each n

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and $\varphi_n(e_{i,j}) = \frac{1}{n}\lambda_{i-j}$.

Systems from Følner sequences

For a countable, discrete, amenable group Γ, we can use any Følner sequence $(\mathcal{G}_n)_n$ to construct a system of ucp approximations of $\mathrm{C}^*_\lambda(\Gamma)$:

$$\begin{tikzcd}
\mathrm{C}^*_\lambda(\Gamma) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_0"] & & \mathrm{C}^*_\lambda(\Gamma) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_1"] & & \mathrm{C}^*_\lambda(\Gamma) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_2"] & & \dots \\
& M_{\mathcal{G}_0} \arrow[ur, "\varphi_0"] & & M_{\mathcal{G}_1} \arrow[ur, "\varphi_1"] & & M_{\mathcal{G}_2} \arrow[ur, "\varphi_2"] & \dots
\end{tikzcd}$$

Identifying $M_{\mathcal{G}_n} \cong P_n B(\ell^2(\Gamma)) P_n$ with $P_n = \mathrm{proj}_{\mathrm{span}\{\delta_g | g \in \mathcal{G}_n\}}$, we set

$$\psi_n(x) = P_n x P_n \text{ for } x \in \mathrm{C}^*_\lambda(\Gamma) \subset B(\ell^2(\Gamma))$$

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Proposition (C.)

If Γ has a non-torsion element (e.g. $\Gamma = \mathbb{Z}$), then the maps $(\psi_n)_n$ will not be approximately multiplicative/ order zero, and the resulting cpc system $(M_{\mathcal{G}_n}, \psi_{n+1} \circ \varphi_n)_n$ will neither be NF nor CPC^*.

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and $\varphi_n(e_{i,j}) = \frac{1}{n}\lambda_{i-j}$. For the bilateral shift λ_1,

$$\|\psi_n(\lambda_1^*\lambda_1) - \psi_n(\lambda_1)^*\psi_n(\lambda_1)\| = \|e_{n,n}\| = 1.$$

Back to our motivating observations

(Asymptotically/Approximately) multiplicative/ order zero maps carry significantly more structure than generic cpc maps. But these can be hard to get our hands on.

Though systems $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations with $(\psi_n)_n$ approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC^*-systems.

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However, we saw that any system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations produces (after possibly passing to a summable subsystem) a cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ whose limit is completely order isomorphic to a nuclear C^*-algebra.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

C^*-encoding systems (C.)

Definition (C.'23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is C^*-encoding if for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n, j > M$

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,j}\big(\rho_{j,k}(x)\rho_{j,k}(y)\big)\| < \varepsilon.$$

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ is completely order isomorphic to the C^*-algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$ with

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \ \forall\ k \geq 0, x, y \in F_k.$$

All together

Definition (Blackadar–Kirchberg '97)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is NF if
$\forall\ k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, $\exists\ M > k$ so that $\forall\ m > n, j > M$

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Definition (C.–Winter '23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is CPC^* if
$\forall\ k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, $\exists\ M > k$ so that $\forall\ m > n, j > M$

$$\|\rho_{m,j}(1_{F_j})\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

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Example (C.)

Any NF system is C^*-encoding, and any CPC^*-system has a C^*-encoding subsystem.

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Example

From our Følner approximation of $\mathrm{C}^*_\lambda(\mathbb{Z})$, the compositions $\rho_{m,n}$ are given on matrix units by (with $S_n \in M_n$ is the shift)

$$\rho_{m,n}(e_{i,j}) = \frac{1}{n}\left(\prod_{k=1}^{m-1} 1 - \frac{|i-j|}{n+k}\right) S_m^{i-j}.$$

Under summability assumptions, this system is C^*-encoding.

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Under summability assumptions, this system is C^*-encoding.

Remark

A unital CPC^* system is automatically NF. This is not so for C^*-encoding systems.

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Theorem (C. '23)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear.*
2. *A is coi to the limit of a C^*-encoding system.*

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Moreover, for any nuclear C^*-algebra A and any[1] *system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations of A the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is C^*-encoding and its limit is coi to A.*

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3. *The system has a C^*-encoding subsystem.*

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3. *The system has a C^*-encoding subsystem.*

That means C^*-encoding is necessary and sufficient to have a limit coi to a (nuclear) C^*-algebra, and that the multiplication

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \ k \geq 0, x, y \in F_k,$$

is essentially the only possible C^*-product on the limit.

Part IV: Nuclear Operator Systems

Non-C^*-encoding systems?

Question

Is there a finite-dimensional cpc system with no C^*-encoding subsystem, i.e., whose limit is not coi to a C^*-algebra?

C^*-encoding systems (C.)

Definition (C.'23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is C^*-encoding if for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n, j > M$

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,j}\big(\rho_{j,k}(x)\rho_{j,k}(y)\big)\| < \varepsilon.$$

A simple example

Example (C.–Galke–van Luijk–Stottmeister)

The finite-dimensional cpc system $(\mathrm{M}_n, \rho_{n+1,n})_n$ with

$$\rho_{n+1,n}(y) = y \oplus y_{11}$$

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Note that for $E_{12}, E_{21} \in \mathrm{M}_2$ and $n > 2$, $\rho_{n,2}(E_{12}) = E_{12} \in \mathrm{M}_n$ and $\rho_{n,2}(E_{21}) = E_{12} \in \mathrm{M}_n$.

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Then for all $m > n > j > 2$

$$\|\rho_{m,n}(\rho_{n,2}(E_{12})\rho_{n,2}(E_{21})) - \rho_{m,j}(\rho_{j,2}(E_{12})\rho_{j,2}(E_{21}))\| = 1.$$

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Yes. Moreover, since the maps in our example are ucp, the limit is an operator system, which is not coi to a C^*-algebra.

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Yes. Moreover, since the maps in our example are ucp, the limit is an operator system, which is not coi to a C^*-algebra. Moreover, it is a nuclear operator system.

Nuclear Operator Systems

Theorem (Han–Paulsen, '11)

A separable operator system $\mathcal{S}$ is nuclear in the category of operator systems (i.e., (max,min)-nuclear) iff there exist ucp maps $\mathcal{S} \xrightarrow{\psi_n} \mathrm{M}_{k_n} \xrightarrow{\varphi_n} \mathcal{S}$ such that $\varphi_n \circ \psi_n \to \mathrm{id}_{\mathcal{S}}$ pointwise in norm.

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Proposition (C.–Galke–van Luijk–Stottmeister)

Let $\mathcal{S}$ be a separable nuclear operator system and $(\mathcal{S} \xrightarrow{\psi_n} \mathrm{M}_{k_n} \xrightarrow{\varphi_n} \mathcal{S})_n$ a system of completely positive approximations. After possibly passing to a summable subsystem, $(\mathrm{M}_{k_n}, \psi_{n+1} \circ \varphi_n)_n$ is a cpc system whose limit is coi to $\mathcal{S}$ via the map $a \mapsto [(\psi_n(a))_n]$.

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Hence any separable nuclear operator system is coi to the limit of a finite-dimensional cpc system.

Nuclear Operator Systems not coi to C^*-algebras

There are relatively few examples of nuclear operator systems which are not coi to C^*-algebras.

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A separable nuclear operator system that does not embed completely order isometrically into a nuclear C^*-algebra.

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Theorem (C.–Galke–van Luijk–Stottmeister)

Let $\mathcal{S}$ be a separable operator system. Then the following are equivalent.

1. *$\mathcal{S}$ is nuclear and completely order isomorphic to a C^*-algebra.*
2. *$\mathcal{S}$ is completely order isomorphic to the limit of a finite-dimensional C^*-encoding system.*

A simple example revisited

Consider the finite-dimensional cpc system $(\mathrm{M}_n, \rho_{n+1,n})_n$ with

$$\rho_{n+1,n}(y) = y \oplus y_{11}.$$

Proposition (C.–Galke–van Luijk–Stottmeister + Han–Paulsen)

The limit is coi to the nuclear operator system

$$\mathcal{S} = \overline{span}\{I, E_{i,j} \mid (i,j) \neq (1,1)\} \subset B(\ell^2(\mathbb{N})).$$

With this and the previous theorem, we recover Han and Paulsen's result.

Theorem (Han–Paulsen, '11)

$\mathcal{S}$ is not coi to a C^*-algebra.

Thank you.

Summability

A system of c.p.c. approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of a separable C^*-algebra A is summable if there exists a decreasing sequence $(\varepsilon_n) \in \ell^1(\mathbb{N})^1_+$ so that $\|\varphi_n - \varphi_m \circ \psi_m \circ \varphi_n\| < \epsilon_n$ for all $m > n \geq 0$.

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We will call a Følner sequence $(\mathcal{G}_n)_n$ for a discrete group G summable if there exists a decreasing sequence $(\varepsilon_n) \in \ell^1(\mathbb{N})^1_+$ so that for all $m > n \geq 0$

$$\max_{g,h \in \mathcal{G}_n} \left(1 - \frac{|\mathcal{G}_m \cap gh^{-1}\mathcal{G}_m|}{|\mathcal{G}_m|}\right) |\mathcal{G}_n| < \varepsilon_m.$$

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One sub-Følner sequence of $(\{0, ..., n\})_n$ for $\mathbb{Z}$ making the system of cpc approximations from before summable (for $\varepsilon_n = 2^{n+1}$) is given by $\mathcal{G}_0 = \{0\}$ and $\mathcal{G}_n = \{0, ..., 2^n|\mathcal{G}_{n-1}|\}$ for $n \geq 1$.

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$$\varphi_n(\psi_n(\lambda_k)) = \varphi_n(S^k_{|\mathcal{G}_n|}) = \frac{|\mathcal{G}_n| - |k|}{|\mathcal{G}_n|}\lambda_k$$

for $n > k \geq 0$ where $S_{|\mathcal{G}_n|} \in M_{|\mathcal{G}_n|}$ is the shift. A few iterations yields

$$\rho_{m,n}(e_{i,j}) = \frac{1}{|\mathcal{G}_n|}\left(\prod_{k=1}^{m-1} \frac{|\mathcal{G}_{n+k}| - |i-j|}{|\mathcal{G}_{n+k}|}\right) S^{i-j}_{|\mathcal{G}_m|}, \quad \text{for } m > n \geq 0, 0 \leq i,j \leq n.$$