

# Nuclear $C^*$ -algebras and generalized inductive limits

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based in part on joint work with W. Winter  
and with N. Galke, L. van Luijk, and A. Stottmeister

UC Berkeley Probabilistic Operator Algebra Seminar

## Part I: Inductive limits and AF algebras

## Inductive limits of $C^*$ -algebras

An **inductive system** of  $C^*$ -algebras consists of a sequence  $(A_n)_n$  of  $C^*$ -algebras together with connecting  $*$ -homomorphisms

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The **inductive limit** of the system  $(A_n, \rho_{m,n})$  is the  $C^*$ -algebra

$$\varinjlim (A_n, \rho_{m,n}) := \overline{\bigcup_k \rho_k(A_k)} \subset \prod A_n / \bigoplus A_n.$$

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Alternatively, an AF  $C^*$ -algebra is one that contains an ascending sequence of finite-dimensional subalgebras with norm-dense union, which makes its von Neumann analogue a little more apparent.

# AFD von Neumann Algebras

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The hyperfinite II<sub>1</sub>-factor  $\mathcal{R}$ .

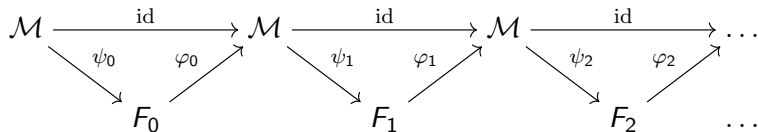
$$M_2 \xrightarrow{a \mapsto a \oplus a} M_4 \hookrightarrow \dots \hookrightarrow \overline{\bigcup_k M_{2^k}}^{wk^*} = \mathcal{R}$$

# Semi-discrete von Neumann Algebras

## Definition

A separably acting von Neumann algebra  $\mathcal{M}$  is **semi-discrete** iff there exists a sequence of finite-dimensional von Neumann algebras  $(F_n)_{n \in \mathbb{N}}$  and unital completely positive (ucp) maps  $\mathcal{M} \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} \mathcal{M}$  such that  $\varphi_n \circ \psi_n \rightarrow \text{id}_{\mathcal{M}}$  pointwise wk\*.

This gives a sequence of (wk\*-)approximately commuting diagrams.

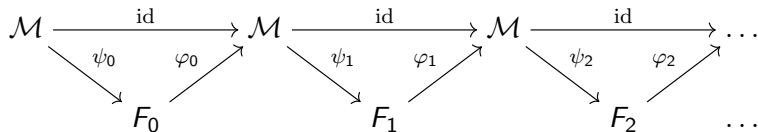


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## Example

Any AFD von Neumann algebra is semi-discrete.

# Nuclear $C^*$ -algebras

Theorem/Definition (Choi–Effros '78; Kirchberg '77)

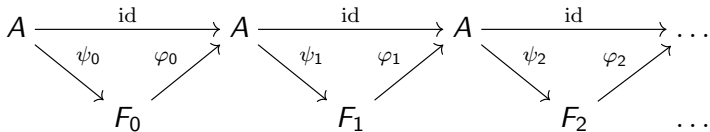
A separable  $C^*$ -algebra  $A$  is **nuclear** iff there exists a sequence of finite-dimensional  $C^*$ -algebras  $(F_n)_{n \in \mathbb{N}}$  and completely positive contractive (cpc) maps  $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$  such that  $\varphi_n \circ \psi_n \rightarrow \text{id}_A$  pointwise in norm.

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We call  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  a **system of cpc approximations** of  $A$ .

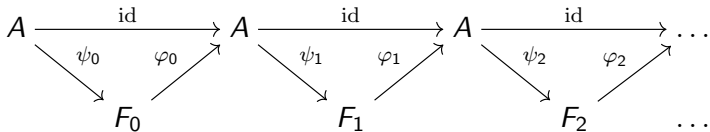


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**Example**

Any AF  $C^*$ -algebra is nuclear.

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## Example

- $C(X)$  where  $\dim(X) \geq 1$ .
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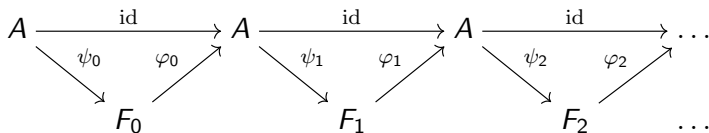
Even though a direct analogue to Connes' result, i.e., “nuclear  $\Rightarrow$  AF”, is out of the question, any system of cpc approximations of a nuclear  $C^*$ -algebra gives rise to something very much like an inductive system.

## From CPAP to an “inductive system”

Theorem/Definition (Choi–Effros '78; Kirchberg '77)

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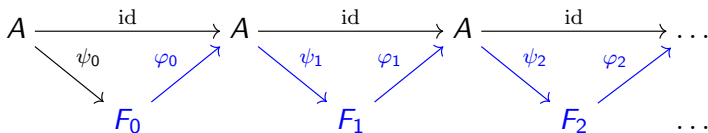


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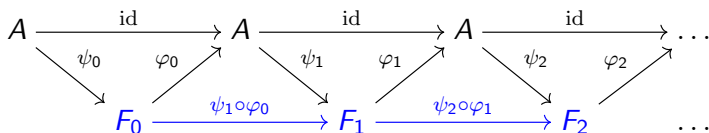


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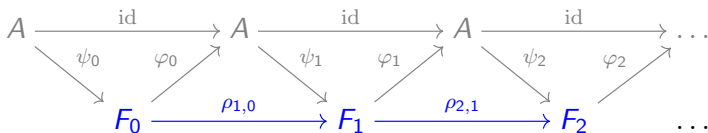


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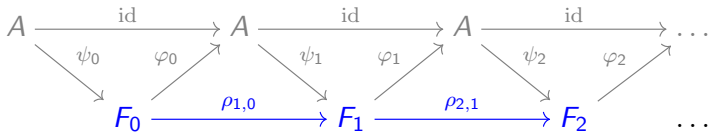
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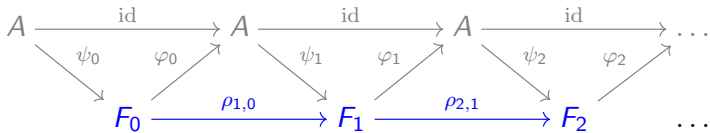
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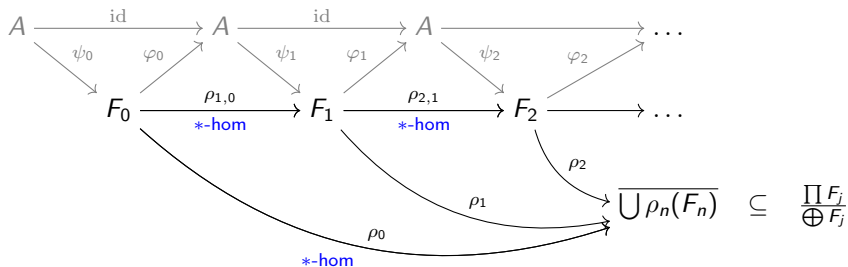
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Forming the limit

Forming the limit with  $*$ -homomorphisms

## Forming the limit with $*$ -homomorphisms

Suppose  $A = \overline{\bigcup F_n}$  with  $\varphi_n : F_n \hookrightarrow A$  the inclusion and  $\psi_n : A \rightarrow F_n$  a conditional expectation. Then the  $\rho_{n+1,n}$  are  $*$ -homomorphisms,

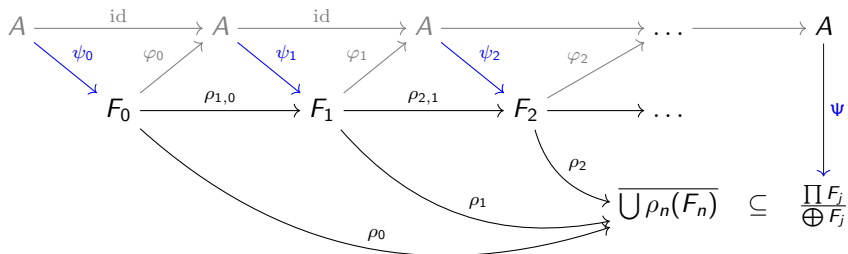


and  $(F_n, \rho_{n+1,n})_n$  is an inductive system with limit

$$\varinjlim (F_n, \rho_{n+1,n}) := \overline{\bigcup \rho_n(F_n)} \subseteq \prod_{\oplus} F_j.$$

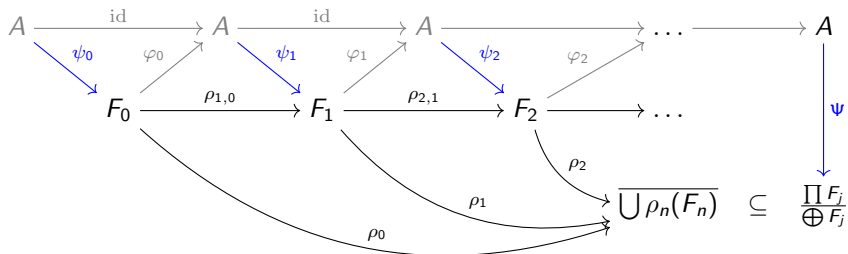
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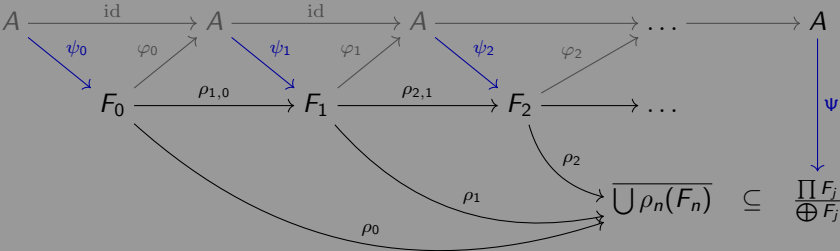
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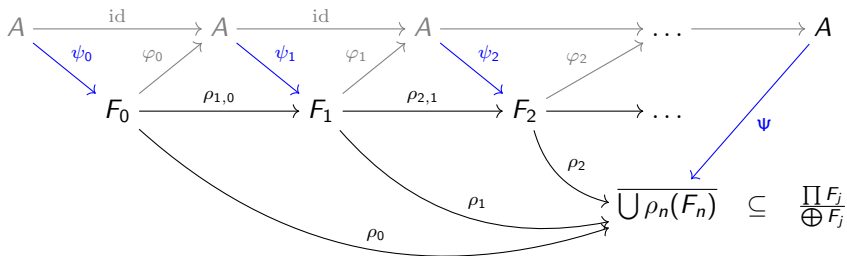
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$$\|\psi_n(a)\| \xrightarrow{n \rightarrow \infty} \|a\|, \quad \forall a \in A$$

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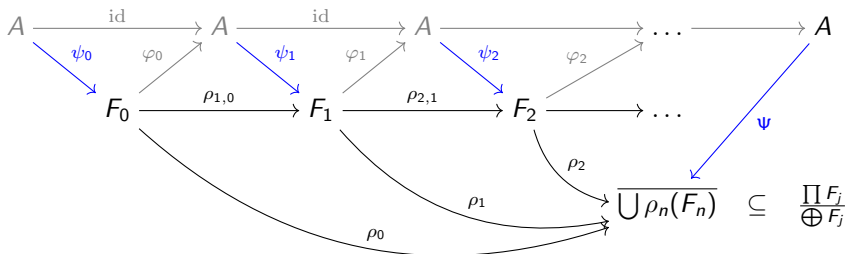
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$$\Psi(A) = \overline{\bigcup \rho_n(F_n)}$$

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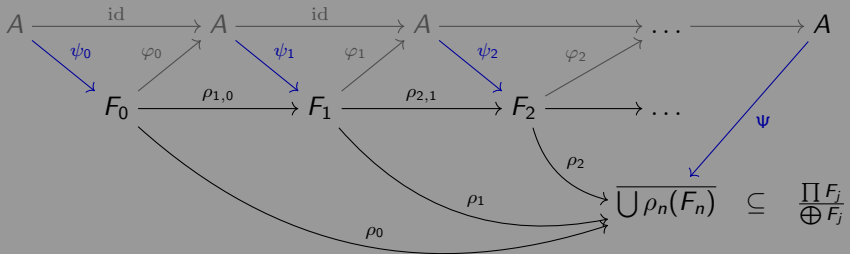
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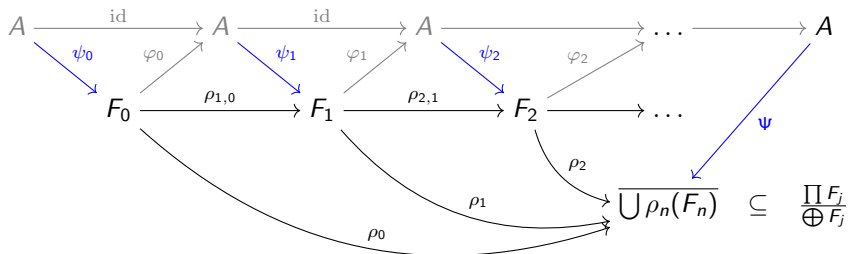
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$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \xrightarrow{n \rightarrow \infty} 0, \forall a, b \in A$$

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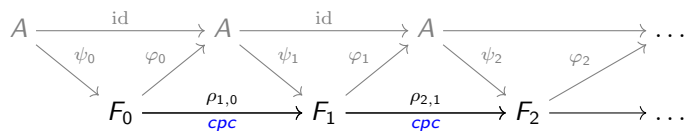
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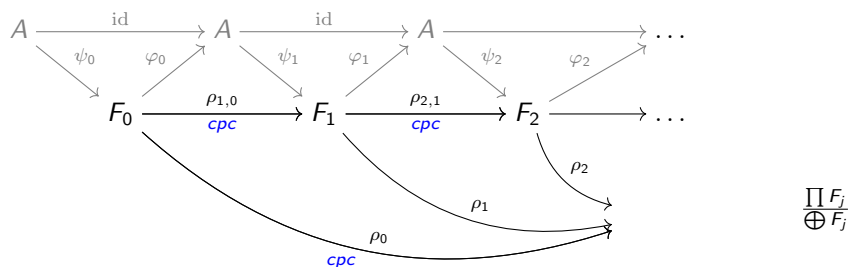
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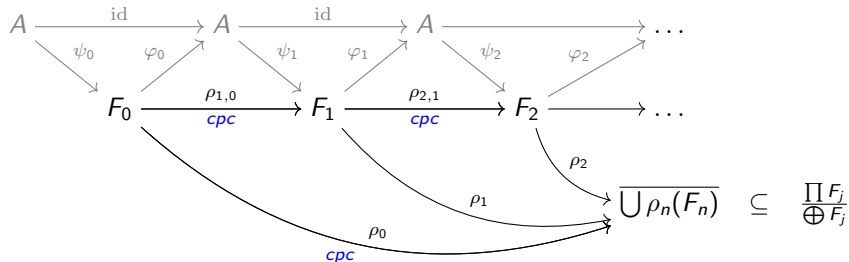
## Forming the limit with cpc maps

When the  $\rho_{n+1,n}$  are **cpc** maps, they still induce **cpc** maps  $\rho_n : F_n \rightarrow \prod F_j / \bigoplus F_j$  with  $\rho_n(x) = [(\rho_{m,n}(x))_{m>n}]$ .



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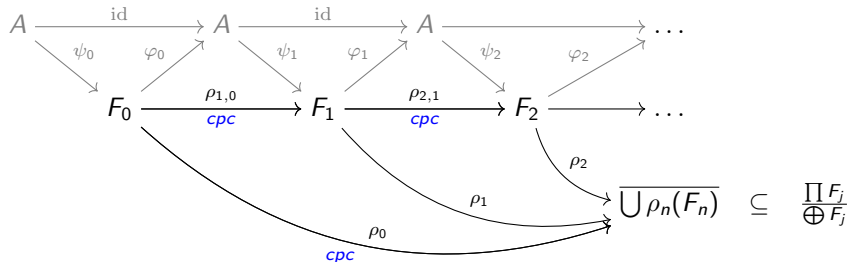
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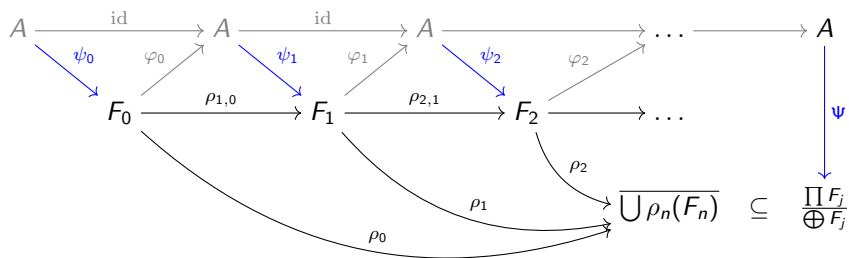
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How does it relate to  $A$ ?

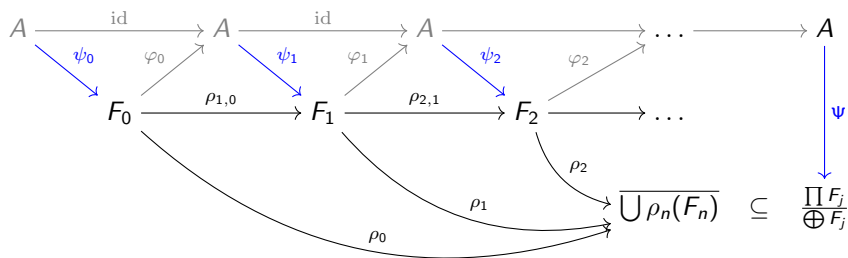
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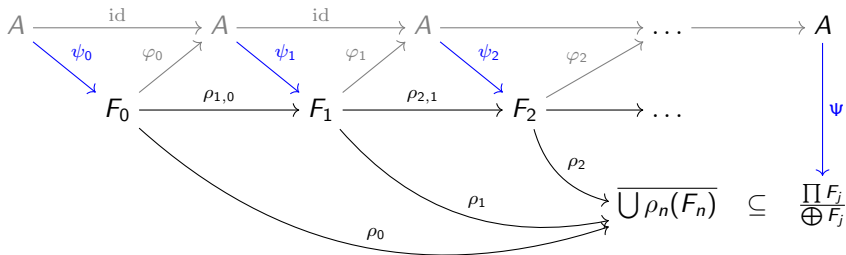


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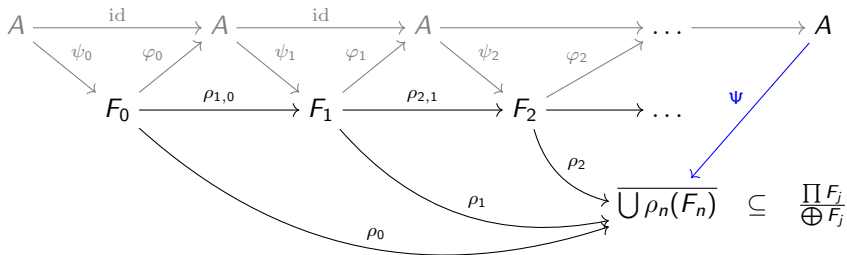
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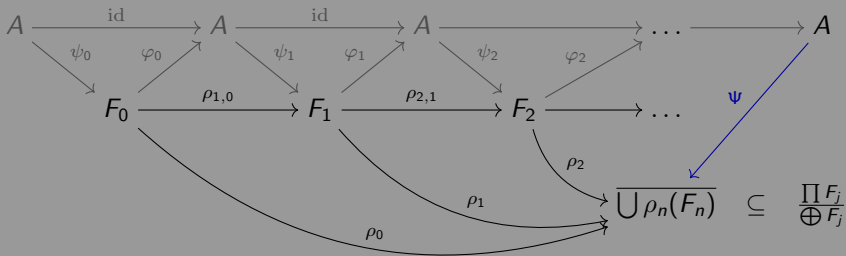


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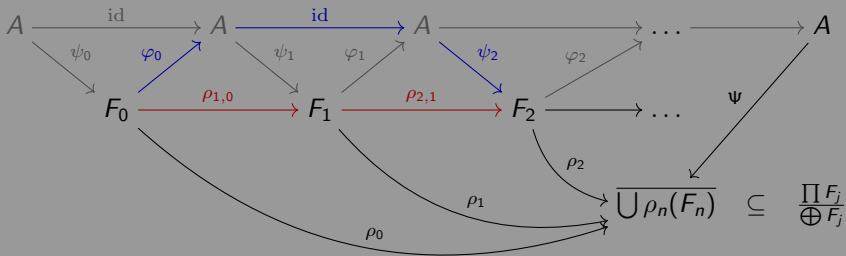
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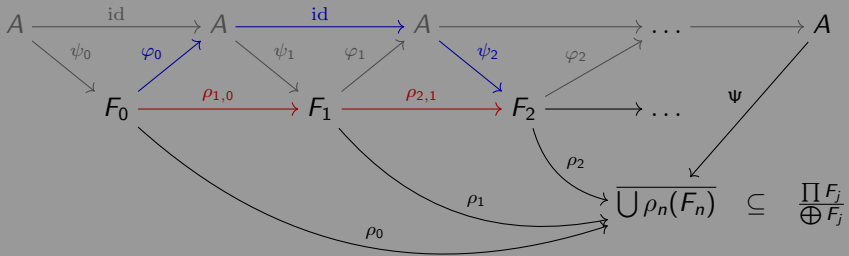
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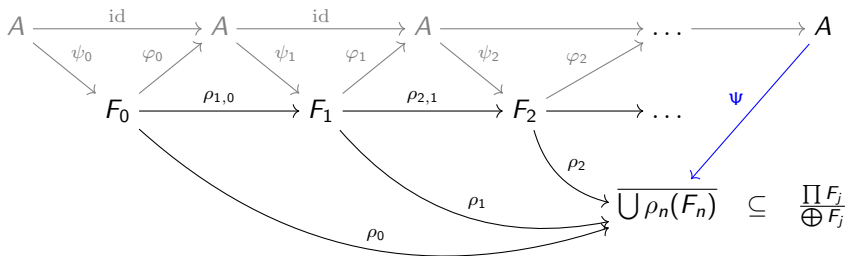
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Any system of cpc approximations admits a summable subsystem, so we assume our system is summable.



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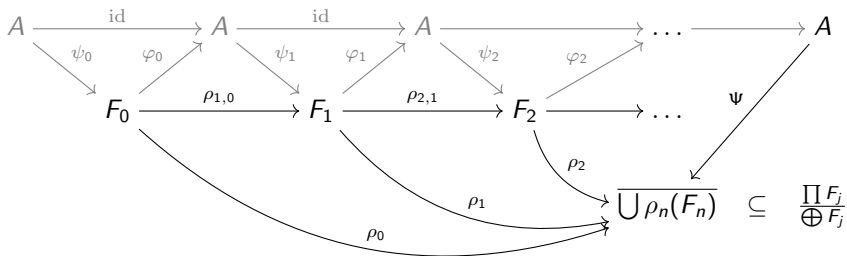


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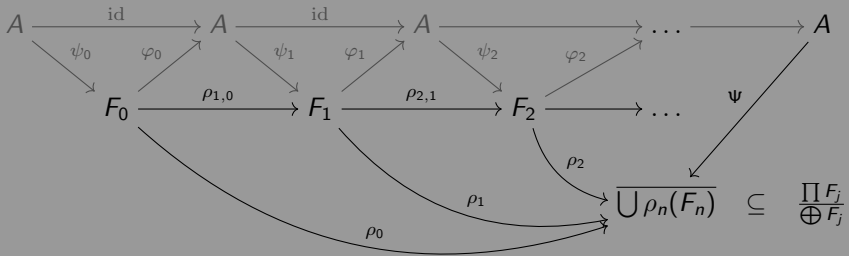
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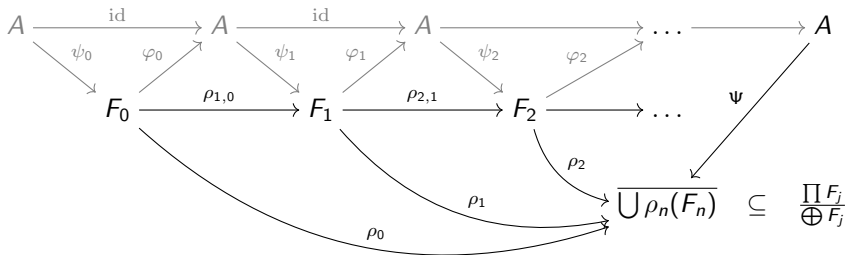
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This can only happen if  $A$  is quasidiagonal (QD).

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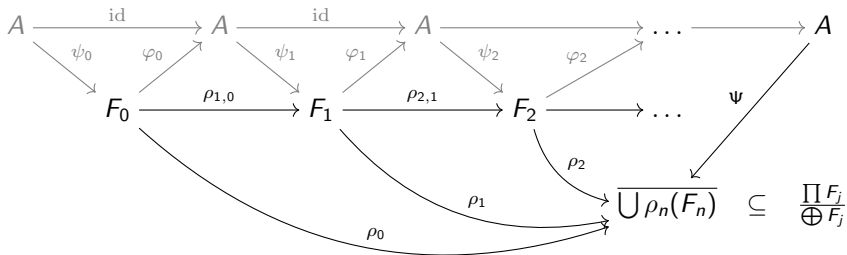


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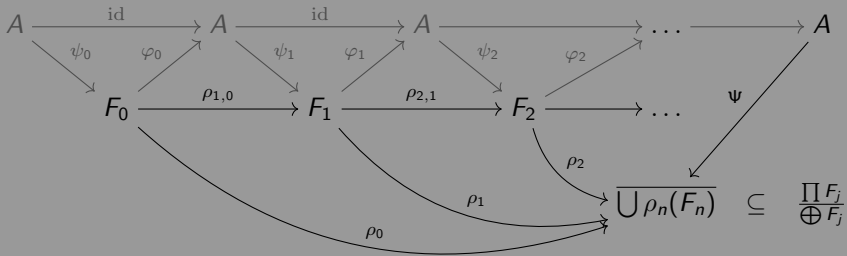
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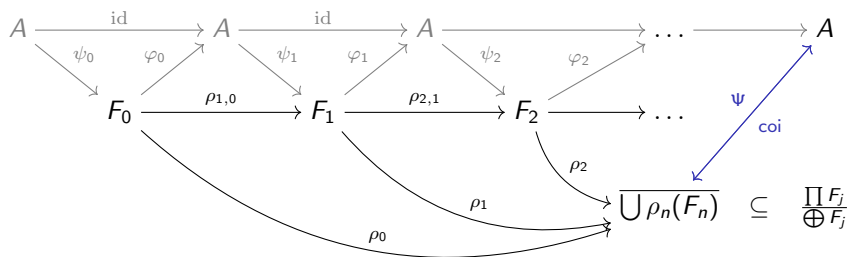
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That means  $\Psi$  is completely isometric and cp with cp inverse.

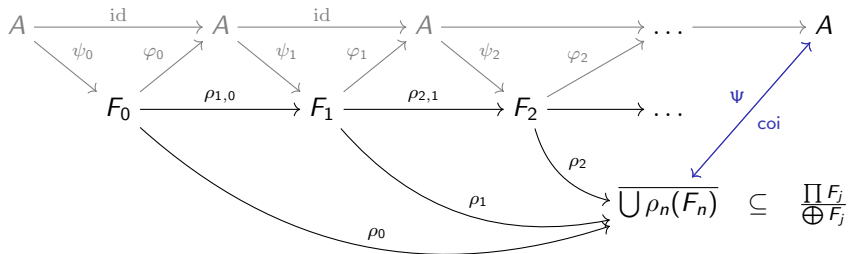
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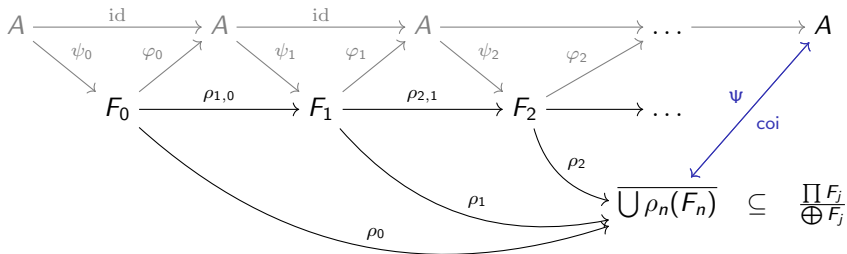


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Moreover, by equipping  $\overline{\bigcup \rho_n(F_n)}$  with the product

$$\Psi(a) \bullet \Psi(b) := \Psi(ab), \quad \forall a, b \in A,$$

we get a  $C^*$ -algebra  $(\overline{\bigcup \rho_n(F_n)}, \bullet)$ , which is  $*$ -isomorphic to  $A$ .

## Part II: cpc systems and nuclearity

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We call a sequence of  $C^*$ -algebras  $(A_n)_n$  together with cpc connecting maps  $\rho_{n+1,n} : A_n \rightarrow A_{n+1}$  a **cpc system**, denoted  $(A_n, \rho_{n+1,n})_n$ . When the  $A_n$  are all finite-dimensional, we call the system **finite-dimensional**.

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In one sense this is a special case of Blackadar and Kirchberg's Generalized Inductive Systems. In another sense, it is a generalization.

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### Question

Given a finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$ , when is the limit  $\overline{\bigcup \rho_n(F_n)}$  coi to a (nuclear)  $C^*$ -algebra?

# Nuclearity

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This follows readily from Ozawa and Sato's One-Way-CPAP, which allows one to determine whether a given  $C^*$ -algebra  $A$  is nuclear by finding a certain family of cpc maps  $\{\varphi_\lambda : F_\lambda \rightarrow A\}_\lambda$  from finite-dimensional  $C^*$ -algebras.



# One Way CPAP

Theorem (Ozawa '02, Sato '21)

A  $C^*$ -algebra  $A$  is nuclear iff there exists a net  $(\varphi_\lambda : F_\lambda \rightarrow A)_{\lambda \in \Lambda}$  of cpc maps from finite-dimensional  $C^*$ -algebras such that the induced cpc map

$$\begin{array}{ccc} \prod_\lambda F_\lambda & \xrightarrow{(\varphi_\lambda)_\lambda} & \ell^\infty(\Lambda, A) \\ \downarrow & & \downarrow \\ \prod_\lambda F_\lambda / \bigoplus_\lambda F_\lambda & \xrightarrow{\Phi} & \ell^\infty(\Lambda, A) / c_0(\Lambda, A) \end{array} \quad \text{satisfies } A^1 \subset \Phi \left( \left( \frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda} \right)^1 \right).$$

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To get the  $\varphi_n$  in our case:

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$\varphi_0$  (dashed blue arrow from  $F_0$  to  $A$ )

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Think of this as saying that for  $m > n > M$ , the maps  $\rho_{m,n}$  become more multiplicative on  $\rho_{n,k}(x)$  and  $\rho_{n,k}(y)$ .

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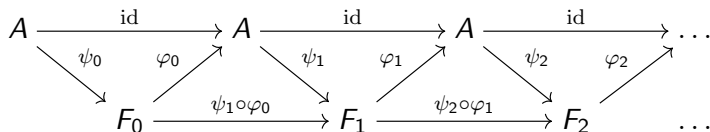
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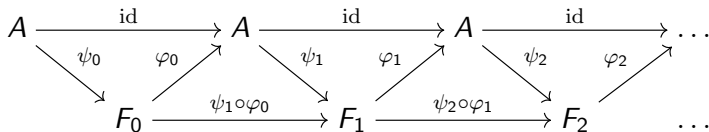
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## Remark

For any summable system of cpc approximations  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ ,  $(\psi_n)_n$  are approximately multiplicative iff  $(F_n, \psi_{n+1} \circ \varphi_n)_n$  is NF.

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The limit  $\overline{\bigcup \rho_n(F_n)} \subset \overline{\prod_{F_j} / \bigoplus_{F_j}}$  of a **CPC\*** system is **completely order isomorphic to the C\*-algebra  $(\overline{\bigcup \rho_n(F_n)}, \bullet)$**  with

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \quad \forall k \geq 0, x, y \in F_k.$$

## Side-by-side

### Definition (Blackadar–Kirchberg '97)

A finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$  is **NF** if

$\forall k \geq 0, x, y \in F_k$ , and  $\varepsilon > 0, \exists M > k$  so that  $\forall m > n, j > M$

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## NF and CPC\*-systems

Theorem (Blackadar–Kirchberg '97)

The following are equivalent for a separable  $C^*$ -algebra  $A$ :

1.  $A$  is *nuclear and QD*.
2.  $A$  is *\*-isomorphic to the limit of an NF system*.

Moreover, for any *nuclear and QD*  $C^*$ -algebra  $A$ , there exists a system  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  with  $(\psi_n)_n$  approximately *multiplicative* so that the induced cpc system  $(F_n, \psi_{n+1} \circ \varphi_n)_n$  is *NF* and its limit is *\*-isom* to  $A$ .

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# Hierarchy

## Example (C.-Winter)

Any NF system has a CPC\*-subsystem.

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## Remark

*If the connecting maps  $\rho_{n+1,n} : F_n \rightarrow F_{n+1}$  are unital, then a CPC\* system is automatically NF.*

## Part III: $C^*$ -encoding systems

## Back to our motivating observations

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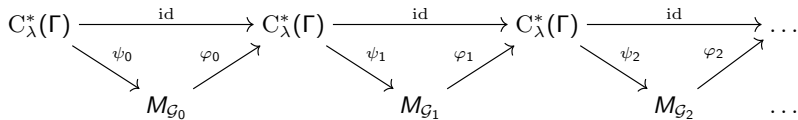
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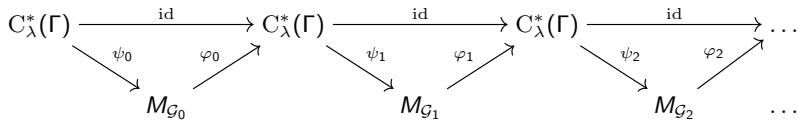
## Systems from Følner sequences

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## Example $C_\lambda^*(\mathbb{Z})$

For  $\Gamma = \mathbb{Z}$  and Følner sets  $\mathcal{G}_n = \{0, \dots, n-1\}$ , we have  $M_{\mathcal{G}_n} = M_n$  with matrix units  $\{e_{i,j}\}_{i,j=0}^{n-1}$ . Then for each  $n$

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 C_\lambda^*(\Gamma) & \xrightarrow{\text{id}} & C_\lambda^*(\Gamma) & \xrightarrow{\text{id}} & C_\lambda^*(\Gamma) & \xrightarrow{\text{id}} & \dots \\
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### Proposition (C.)

If  $\Gamma$  has a non-torsion element (e.g.  $\Gamma = \mathbb{Z}$ ), then the maps  $(\psi_n)_n$  will not be approximately multiplicative/ order zero, and the resulting cpc system  $(M_{\mathcal{G}_n}, \psi_{n+1} \circ \varphi_n)_n$  will neither be NF nor  $CPC^*$ .

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and  $\varphi_n(e_{i,j}) = \frac{1}{n} \lambda_{i-j}$ . For the bilateral shift  $\lambda_1$ ,

$$\|\psi_n(\lambda_1^* \lambda_1) - \psi_n(\lambda_1)^* \psi_n(\lambda_1)\| = \|e_{n,n}\| = 1.$$

## Back to our motivating observations

(Asymptotically/Approximately) multiplicative/ order zero maps carry significantly more structure than generic cpc maps. But these can be hard to get our hands on.

Though systems  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  of cpc approximations with  $(\psi_n)_n$  approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC\*-systems.

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However, we saw that any system  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  of cpc approximations produces (after possibly passing to a summable subsystem) a cpc system  $(F_n, \psi_{n+1} \circ \varphi_n)_n$  whose limit is completely order isomorphic to a nuclear  $C^*$ -algebra.

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### Question

Given a finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$ , when is the limit  $\overline{\bigcup \rho_n(F_n)}$  coi to a (nuclear) C\*-algebra?

## $C^*$ -encoding systems (C.)

### Definition (C.'23)

A finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$  is  $C^*$ -encoding if for any  $k \geq 0$ ,  $x, y \in F_k$ , and  $\varepsilon > 0$ , there exists an  $M > k$  so that for all  $m > n, j > M$

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## All together

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From our Følner approximation of  $C_\lambda^*(\mathbb{Z})$ , the compositions  $\rho_{m,n}$  are given on matrix units by (with  $S_n \in M_n$  is the shift)

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## Remark

*A unital  $CPC^*$  system is automatically NF. This is not so for  $C^*$ -encoding systems.*

## $C^*$ -encoding systems

Theorem (C. '23)

*The following are equivalent for a separable  $C^*$ -algebra  $A$ :*

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<sup>1</sup>after possibly passing to a summable subsystem— same for NF and  $CPC^*$

## $C^*$ -encoding systems

### Theorem (C. '23)

The following are equivalent for a separable  $C^*$ -algebra  $A$ :

1.  $A$  is nuclear.
2.  $A$  is coi to the limit of a  $C^*$ -encoding system.

Moreover, for any nuclear  $C^*$ -algebra  $A$  and *any*<sup>1</sup> system

$(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  of cpc approximations of  $A$  the induced cpc system  $(F_n, \psi_{n+1} \circ \varphi_n)_n$  is  $C^*$ -encoding and its limit is coi to  $A$ .

### Theorem (C.–Winter '23 (via Blackadar-Kirchberg + Voiculescu))

Moreover, for any nuclear  $C^*$ -algebra  $A$ , *there exists* a system

$(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  *with*  $(\psi_n)_n$  *approximately order zero* so that the induced cpc system  $(F_n, \psi_{n+1} \circ \varphi_n)_n$  is  $CPC^*$  and its limit is coi to  $A$ .

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## $C^*$ -encoding systems

### Question

Given a finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$ , when is the limit  $\overline{\bigcup \rho_n(F_n)}$  coi to a (nuclear)  $C^*$ -algebra?

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### Theorem (C. '23)

*For a finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$ , the following are equivalent*

- 1. The limit is coi to a  $C^*$ -algebra.*
- 2. The limit is coi to a nuclear  $C^*$ -algebra. (CW, OS)*
- 3. The system has a  $C^*$ -encoding subsystem.*

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That means  $C^*$ -encoding is necessary and sufficient to have a limit coi to a (nuclear)  $C^*$ -algebra, and that the multiplication

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x) \rho_{n,k}(y)), \quad k \geq 0, x, y \in F_k,$$

is essentially the only possible  $C^*$ -product on the limit.

## Part IV: Nuclear Operator Systems

# Non- $C^*$ -encoding systems?

## Question

*Is there a finite-dimensional cpc system with no  $C^*$ -encoding subsystem, i.e., whose limit is not coi to a  $C^*$ -algebra?*



## $C^*$ -encoding systems (C.)

### Definition (C.'23)

A finite-dimensional cpc system  $(F_n, \rho_{n+1,n})_n$  is  $C^*$ -encoding if for any  $k \geq 0$ ,  $x, y \in F_k$ , and  $\varepsilon > 0$ , there exists an  $M > k$  so that for all  $m > n, j > M$

$$\|\rho_{m,n}(\rho_{n,k}(x)\rho_{n,k}(y)) - \rho_{m,j}(\rho_{j,k}(x)\rho_{j,k}(y))\| < \varepsilon.$$

## A simple example

### Example (C.–Galke–van Luijk–Stottmeister)

The finite-dimensional cpc system  $(M_n, \rho_{n+1,n})_n$  with

$$\rho_{n+1,n}(y) = y \oplus y_{11}$$

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Note that for  $E_{12}, E_{21} \in M_2$  and  $n > 2$ ,  $\rho_{n,2}(E_{12}) = E_{12} \in M_n$  and  $\rho_{n,2}(E_{21}) = E_{12} \in M_n$ .

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Then for all  $m > n > j > 2$

$$\|\rho_{m,n}(\rho_{n,2}(E_{12})\rho_{n,2}(E_{21})) - \rho_{m,j}(\rho_{j,2}(E_{12})\rho_{j,2}(E_{21}))\| = 1.$$

## Non- $C^*$ -encoding systems

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Yes. Moreover, since the maps in our example are ucp, the limit is an operator system, which is not coi to a  $C^*$ -algebra.



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Yes. Moreover, since the maps in our example are ucp, the limit is an operator system, which is not coi to a  $C^*$ -algebra. Moreover, it is a nuclear operator system.

# Nuclear Operator Systems

Theorem (Han–Paulsen, '11)

A separable operator system  $\mathcal{S}$  is *nuclear* in the category of operator systems (i.e., (max,min)-nuclear) iff there exist ucp maps

$\mathcal{S} \xrightarrow{\psi_n} M_{k_n} \xrightarrow{\varphi_n} \mathcal{S}$  such that  $\varphi_n \circ \psi_n \rightarrow \text{id}_{\mathcal{S}}$  pointwise in norm.

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## Proposition (C.–Galke–van Luijk–Stottmeister)

Let  $\mathcal{S}$  be a separable nuclear operator system and  $(\mathcal{S} \xrightarrow{\psi_n} M_{k_n} \xrightarrow{\varphi_n} \mathcal{S})_n$  a system of completely positive approximations. After possibly passing to a summable subsystem,  $(M_{k_n}, \psi_{n+1} \circ \varphi_n)_n$  is a cpc system whose limit is coi to  $\mathcal{S}$  via the map  $a \mapsto [(\psi_n(a))_n]$ .

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Hence any separable nuclear operator system is coi to the limit of a finite-dimensional cpc system.

## Nuclear Operator Systems not cois to $C^*$ -algebras

There are relatively few examples of nuclear operator systems which are not cois to  $C^*$ -algebras.

## Nuclear Operator Systems not coisotropic to $C^*$ -algebras

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**Example (Kirchberg–Wassermann, '98)**

A separable nuclear operator system that does not embed completely order isometrically into a nuclear  $C^*$ -algebra.

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### Example (Kirchberg–Wassermann, '98)

A separable nuclear operator system that does not embed completely order isometrically into a nuclear  $C^*$ -algebra.

### Theorem (C.–Galke–van Luijk–Stottmeister)

*Let  $S$  be a separable operator system. Then the following are equivalent.*

- 1.  $S$  is nuclear and completely order isomorphic to a  $C^*$ -algebra.*
- 2.  $S$  is completely order isomorphic to the limit of a finite-dimensional  $C^*$ -encoding system.*

## A simple example revisited

Consider the finite-dimensional cpc system  $(M_n, \rho_{n+1,n})_n$  with

$$\rho_{n+1,n}(y) = y \oplus y_{11}.$$

Proposition (C.–Galke–van Luijk–Stottmeister + Han–Paulsen)

*The limit is coi to the nuclear operator system*

$$\mathcal{S} = \overline{\text{span}}\{I, E_{ij} \mid (i, j) \neq (1, 1)\} \subset B(\ell^2(\mathbb{N})).$$

With this and the previous theorem, we recover Han and Paulsen's result.

Theorem (Han–Paulsen, '11)

*$\mathcal{S}$  is not coi to a  $C^*$ -algebra.*



Thank you.

## Summability

A system of c.p.c. approximations  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  of a separable  $C^*$ -algebra  $A$  is **summable** if there exists a decreasing sequence  $(\epsilon_n) \in \ell^1(\mathbb{N})_+^1$  so that  $\|\varphi_n - \varphi_m \circ \psi_m \circ \varphi_n\| < \epsilon_n$  for all  $m > n \geq 0$ .

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We will call a Følner sequence  $(\mathcal{G}_n)_n$  for a discrete group  $G$  **summable** if there exists a decreasing sequence  $(\varepsilon_n) \in \ell^1(\mathbb{N})_+^1$  so that for all  $m > n \geq 0$

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One sub-Følner sequence of  $(\{0, \dots, n\})_n$  for  $\mathbb{Z}$  making the system of cpc approximations from before summable (for  $\varepsilon_n = 2^{n+1}$ ) is given by  $\mathcal{G}_0 = \{0\}$  and  $\mathcal{G}_n = \{0, \dots, 2^n |\mathcal{G}_{n-1}|\}$  for  $n \geq 1$ .

## Summability

A system of c.p.c. approximations  $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$  of a separable  $C^*$ -algebra  $A$  is **summable** if there exists a decreasing sequence  $(\varepsilon_n) \in \ell^1(\mathbb{N})_+^1$  so that  $\|\varphi_n - \varphi_m \circ \psi_m \circ \varphi_n\| < \varepsilon_n$  for all  $m > n \geq 0$ .

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$$\varphi_n(\psi_n(\lambda_k)) = \varphi_n(S_{|\mathcal{G}_n|}^k) = \frac{|\mathcal{G}_n| - |k|}{|\mathcal{G}_n|} \lambda_k$$

for  $n > k \geq 0$  where  $S_{|\mathcal{G}_n|} \in M_{|\mathcal{G}_n|}$  is the shift. A few iterations yields

$$\rho_{m,n}(e_{i,j}) = \frac{1}{|\mathcal{G}_n|} \left( \prod_{k=1}^{m-1} \frac{|\mathcal{G}_{n+k}| - |i-j|}{|\mathcal{G}_{n+k}|} \right) S_{|\mathcal{G}_m|}^{i-j}, \quad \text{for } m > n \geq 0, 0 \leq i, j \leq n.$$