Selfless C*-algebras

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February 29, 2024

$\mathcal{Z}$ is a simple, separable, unital, C*-algebra such that

$$K_0(\mathcal{Z}) \cong \mathbb{Z}, \quad K_1(\mathcal{Z}) \cong 0, \quad T(\mathcal{Z}) = \{*\}$$

It can be expressed as an inductive limit of dimension drop C*-algebras

$$Z_{n-1,n} = \{f \in C([0,1], M_{n-1} \otimes M_n) : f(0) \in M_{n-1} \otimes 1_n,\ f(1) \in 1_{n-1} \otimes M_n\}.$$

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Selfabsortion:

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$$

In fact,

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots$$

In fact, $\mathcal{Z}$ is strongly self-absorbing, i.e., there exists an isomorphism $\phi\colon \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ that is a.u. to the embedding $\mathcal{Z} \ni z \mapsto z \otimes 1 \in \mathcal{Z} \otimes \mathcal{Z}$.

Strict comparison of positive elements by traces

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Given a tracial state $\tau \in T(A)$ and $a \in A_+$, define

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \tau(p_a).$$

Cuntz comparison of positive elements: Given positive elements a and b,

$$a \precsim b \text{ if } d_n b d_n^* \to a \text{ for some } (d_n)_{n=1}^\infty.$$

A simple unital C*-algebra A has *strict comparison of positive elements by traces* if

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(A) \Rightarrow a \precsim b,$$

for all positive $a, b \in \bigcup_{n=1}^\infty M_n(A)$.

A is called $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$.

Theorem (Rordam)

If A is $\mathcal{Z}$-stable, then it has strict comparison of positive elements by 2-quasitraces (by traces, if A is exact).

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Conjecture (Toms-Winter)

A simple, separable, nuclear C-algebra with strict comparison is $\mathcal{Z}$-stable.*

Currently, this is known to hold if $\partial_e T(A)$ is closed and has finite covering dimension (in particular, in the unique trace case).

Let A be a unital C*-algebra. Let $\tau \in T(A)$ be a faithful tracial state.

Lemma

If $p, q \in A$ are freely independent projections such that $\tau(p) < \tau(q)$, then $upu^* \leq q$ for some unitary u.

Proof.

Calculation of distribution of $p(1-q)p$ by Voiculescu; similar result by Anderson-Blackadar-Haagerup; Dykema. □

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The above lemma extends to positive elements:

Lemma

If $a, b \in A$ are freely independent positive elements such that $d_\tau(a) < d_\tau(b)$, then $a \precsim b$.

Proof: Reduces to the case of projections.

Theorem (Rordam)

$C_r^*(F_\infty)$ has strict comparison of positive elements.

Similar results on strict comparison of projection obtained by Dykema and Rordam.
Note: $C_r^*(F_\infty)$ is tensorially prime.

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Note: $C^*_r(F_\infty)$ is tensorially prime.

Proof.

(Sketch) Let $a, b \in C^*_r(F_\infty)$ be positive and such that $d_\tau(a) < d_\tau(b)$.
After perturbations, reduce to the case that a, b are finite linear combinations of $\{u_w : w \in F_\infty\}$.
Find a symbol $g \in F_\infty$ never used in these linear combinations. Then a and $u_g b u_g^*$ are freely independent. Apply lemma relating freeness with strict comparison. □

Let (A, τ) be a C*-algebra with a faithful tracial state. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Denote by $A^{\mathcal{U}}$ the ultrapower of A and by $\tau_{\mathcal{U}}$ the extension of τ to $A^{\mathcal{U}}$.

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Suppose we can find a Haar $u \in A^{\mathcal{U}}$ such that

- A and $C^*(u)$ are freely independent.
- $\tau_{\mathcal{U}}$ is faithful on $C^*(A, u)$.

Then we have strict comparison: If $a, b \in A_+$ are such that $d_\tau(a) < d_\tau(b)$, then

$$d_{\tau_{\mathcal{U}}}(a) < d_{\tau_{\mathcal{U}}}(ubu^*)$$

and a, ubu^* are freely independent. By the lemma relating freeness to strict comparison, $a \precsim ubu^* \sim b$ in $A^{\mathcal{U}}$. Hence $a \precsim b$ in A.

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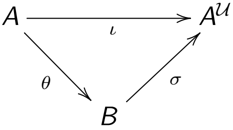
Theorem (Popa)

If M is a separable II_1 factor and $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$ then there exists $u \in M^{\mathcal{U}}$ freely independent from M.

A unital embedding $\theta\colon A \to B$ is called approximately split injective (a.s.i.) if for some free ultrafilter $\mathcal{U}$ there exists $\sigma\colon B \to A^{\mathcal{U}}$ such that $\sigma\theta$ agrees with the diagonal embedding of A in $A^{\mathcal{U}}$:



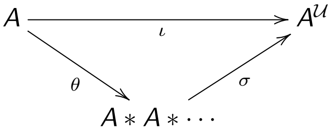
This is equivalent to asking that θ be positively existential, i.e., that for any quantifier-free positive formula $\phi(\bar{x}, \bar{y})$ in the language of unital C*-algebras and tuple $\bar{a}$ in A,

$$\inf_{\bar{y}} \phi(\bar{a}, \bar{y})^A = \inf_{\bar{y}} \phi(\theta(\bar{a}), \bar{y})^B.$$

Let (A, τ) be a C*-probability space, with τ faithful tracial state.

Definition

(A, τ) is called selfless if the embedding of (A, τ) into the first factor of $(A, \tau) * (A, \tau) * \cdots$ is a.s.i.



Lemma: If $A \neq \mathbb{C}$, then $*_{i=1}^{\infty} A$ contains a Haar unitary.

Theorem

Let (A, τ) be a C-probability space, with τ a faithful trace and $A \neq \mathbb{C}$. TFAE:*

1. *(A, τ) is selfless.*
2. *The embedding of (A, τ) in $(A, \tau) * (C(\mathbb{T}), \lambda)$ is a.s.i.*
3. *The embedding of (A, τ) in $(A, \tau) * (C_r^*(F_\infty), \rho)$ is a.s.i.*

Theorem

Let (A, τ) be selfless, with $A \neq \mathbb{C}$. Then

1. *A is an infinite dimensional simple C*-algebra of stable rank one,*
2. *τ is the unique tracial state, and unique 2-quasitracial state, of A,*
3. *A has the uniform Dixmier property and strict comparison of positive elements with respect to τ.*

Proof.

These properties are true for $A * A * \cdots$, and get passed on to A via the factorization of $A \hookrightarrow A^{\mathcal{U}}$ through $A * A * \cdots$. □

Theorem

Let $(A_i)_{i\in I}$ be an upward directed family of subalgebras of $A = \overline{\bigcup_{i\in I} A_i}$. If $(A_i, \tau|_{A_i})$ is selfless for all i, then (A, τ) is selfless.

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Theorem

Let (A, τ) be selfless, with $A \neq \mathbb{C}$. If A' is a unital C-algebra stably isomorphic to A, then (A', τ') is again selfless, where τ' denotes the unique tracial state on A'.*

Note: $A' \cong pM_n(A)p$, with p projection.

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Theorem

Let $(A_i, \tau_i)_{i\in I}$, with I an infinite set, be C-probability spaces with τ_i a faithful trace for all i. Suppose that, for infinitely many i, τ_i vanishes on some unitary of A_i. Then $(A, \tau) = *_{i\in I}(A_i, \tau_i)$ is selfless.*

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The following C-algebras are selfless:*

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Proof.

(2): Ozawa showed that $C^*_r(F_\infty) \hookrightarrow \mathcal{Z}^{\mathcal{U}}$ (building on $C^*_r(F_\infty)$ being MF).
On the other hand $\mathcal{Z} \hookrightarrow C^*_r(F_\infty)$ (using classification by the Cuntz semigroup, and that $C^*_r(F_\infty)$ has stable rank one and strict comparison), Combining these results

$$\mathcal{Z} * C(\mathbb{T}) \hookrightarrow \mathcal{Z}^{\mathcal{U}}.$$

But all embeddings $\mathcal{Z} \hookrightarrow \mathcal{Z}^{\mathcal{U}}$ are unitarily equivalent.
(3) UHF case: $M_{2^\infty} = \lim M_{2^n}(\mathcal{Z})$. □

Definition (Dykema, Rordam)

A C*-probability space (A, τ) is *eigenfree* if there exist an endomorphism $\theta\colon A \to A$ and a Haar unitary $u \in A$ such that $\theta(A)$ and $C^*(u)$ are freely independent and $\tau\theta = \tau$.

Example: $C_r^*(F_2)$ is eigenfree.

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Example: $C_r^*(F_2)$ is eigenfree.

Theorem

Let (A, τ) be a C-probability space, with τ a faithful trace. Suppose that (A, τ) is eigenfree relative to an endomorphism $\theta\colon A \to A$. Let B be the inductive limit of the stationary system $A \xrightarrow{\theta} A$ and $\bar{\tau}$ the trace on B, projective limit of the trace τ. Then $(B, \bar{\tau})$ is selfless.*

Question

How prevalent is selflessness?

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Is $C_r^*(F_2)$ selfless?

What this entails: Let $F_2 = \langle a, b \rangle$ and $F_3 = \langle a, b, c \rangle$. Then we seek a Haar unitary $u \in C_r^*(F_2)^{\mathcal{U}}$ such that $a \mapsto a$, $b \to b$, and $c \mapsto u$ extends to an isomorphism $C_r^*(F_3) \cong C^*(a, b, u)$.

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Question

Does $C_r^*(F_2)$ have strict comparison?

There are many known structural properties of C*-algebras with strict comparison that we cannot presently verify for $C_r^*(F_2)$.

Question

Is any trace zero element in $C_r^*(F_2)$ a sum of at most 3 (or any other bound) commutators?

Question

If (A, τ) is selfless and (B, ρ) a C-probability space with ρ a faithful trace, is $(A * B, \tau * \rho)$ selfless?*

Say $u \in A^{\mathcal{U}}$ is a Haar unitary such that $C^*(A, u) \cong A * C(\mathbb{T})$. Is $(\tau * \rho)_{\mathcal{U}}$ faithful on $C^*(A * B, u) \subseteq (A * B)^{\mathcal{U}}$?

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Definition

Let (A, ρ) be a C*-probability space, where ρ induces a faithful GNS representation. Set $(B, \bar{\rho}) = *_{i=1}^{\infty}(A, \rho)$. We call (A, ρ) selfless if for the embedding $\theta \colon (A, \rho) \to (B, \bar{\rho})$ into the first factor there exists an ultrafilter $\mathcal{U}$ and a homomorphism $\sigma \colon B \to A^{\mathcal{U}}$ such that $\rho_{\mathcal{U}}\sigma = \bar{\rho}$ and $\sigma\theta$ agrees with the diagonal embedding of A in $A^{\mathcal{U}}$.

Thank you!