Free probability of type B prime*

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Backgrounds

Let X be an $N \times N$ GUE such that its empirical eigenvalue distribution

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(X)} \to \frac{1}{2\pi}\sqrt{4-x^2}1_{(-2,2)}(x)dx, \qquad N \to \infty.$$

It is also known that $\lambda_1(X) \to 2$ (Strong convergence). Baik, Ben Arous, Peché studied finite-rank perturbations. Especially, Peché (2006) proved

$$\lambda_1(X+\theta E_{11}) \to \begin{cases} 2 & \text{if } \theta \le 1, \\ \theta + 1/\theta, & \text{if } \theta > 1 \text{ (Outlier)}. \end{cases}$$

Some works from free probabilistic perspectives (not comprehensive)

- S. T. Belinschi, H. Bercovici, M. Capitaine, and M. Février, Ann. Probab. 45 (2017), 3571–3625.
- S. T. Belinschi, H. Bercovici, M. Capitaine, Int. Math. Res. Not. 4 (2021), 2588–2641.
- G. Cébron, A. Dahlqvist, and F. Gabriel, arXiv:2205.01926.
- D. Shlyakhtenko, Indiana Univ. Math. J. 67 (2018), no. 2, 971–991.

Aim

We want to describe large-N limit of normalized / unnormalized traces of polynomials in random matrices

$$\{U_iA_iU_i^*, V_iF_iV_i^*\}_{i\in I} \qquad (\text{or} \quad \{U_iA_iU_i^*, F_i\}_{i\in I}) \tag{1}$$

where

- $\{U_i, V_i : i \in I\}$ is independent family of Haar unitary matrices,
- $A_i = A_i^{(N)}$ are deterministic matrices in $\mathrm{M}_N(\mathbb{C})$ such that for each i

$$(U_iA_iU_i^*, \frac{1}{N}\operatorname{Tr}_{\mathrm{M}_N(\mathbb{C})}) \to (a_i, \varphi)$$

  where $\{a_i\}$ are nc r.v.s in a nc probability space $(\mathcal{A}, \varphi)$.
- F_i are deterministic matrices such that, for each i,

$$(V_iF_iV_i^*, \operatorname{Tr}_{\mathrm{M}_N(\mathbb{C})}) \to (f_i, \Phi)$$

  where $\{f_i\}$ are nc r.v.s in a nc probability space $(\mathcal{F}, \Phi)$.

Question: How to describe the mixtures of a_i and f_i, e.g. the limit of $\operatorname{Tr}(U_1A_1U_1^*V_1F_1V_1^*U_2A_2U_2^*)$?

Aim (continued)

a_i: the limit of $U_i A_i U_i^*$ (main part), φ: the limit of $\frac{1}{N}\operatorname{Tr}_N$
f_i: the limit of $V_i F_i V_i^*$ (perturbation part), Φ: the limit of Tr_N

Known results

- the family $\{U_i A_i U_i^*\}_{i\in I}$ is asymptotically free (Voiculescu), i.e., $\{a_i\}$ is free in $(\mathcal{A},\varphi)$. For example,

$$\varphi(a_1a_2a_1a_2) = \varphi(a_1^2)\varphi(a_2)^2 + \varphi(a_1)^2\varphi(a_2^2) - \varphi(a_1)^2\varphi(a_2)^2.$$

- the pair $(\langle U_iA_iU_i^* : i\in I\rangle, \langle V_iF_iV_i^* : i \in I\rangle)$ is asymptotically cyclic-antimonotone indep. (Collins, Hasebe, Sakuma '18, cf. Shlyakhtenko '18), for example

$$\text{“}\Phi(a_1f_1a_2f_2a_3) = \Phi(f_1f_2)\varphi(a_1a_3)\varphi(a_2)\text{”}$$

- the family $\{V_iF_iV_i^* : i \in I\}$ is asympt. trivially indep. (Collins, Hasebe, Sakuma '18), i.e., all (genuine) mixed moments vanish, e.g.

$$\Phi(f_1f_2) = \Phi(f_1f_2f_1) = 0, \qquad f_1\in\mathcal{F}_1, f_2\in\mathcal{F}_2$$

These rules allow us to calculate any traces of any polynomials.

An abstract framework

The following framework allows us to describe the previous limits very well.

Definition

1. Let $(\mathcal{A}, \varphi)$ be a unital ncps.
2. $(\mathcal{F}, \Phi)$: ncps, where $\mathcal{F}$ is an $\mathcal{A}$-**algebra**, i.e., $\mathcal{F}$ is an algebra having an $\mathcal{A}$-bimodule structure consistent with $\mathcal{F}$'s own multiplication.

The tuple $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$ is called a **ncps of type B$'$**.

We further set

$$\mathcal{B} \equiv \mathcal{A}\langle\mathcal{F}\rangle := \mathcal{A} \oplus \mathcal{F}, \tag{2}$$

$$(a_1, f_1) \cdot (a_2, f_2) := (a_1 a_2, a_1 f_2 + f_1 a_2 + f_1 f_2) \tag{3}$$

$$\varphi(a + f) := \varphi(a), \qquad \underline{\varphi'(a + f) := \Phi(f)}. \tag{4}$$

We thus get a triple $(\mathcal{B}, \varphi, \varphi')$ having two linear functionals. (Infinitesimal nc prob. space)

An abstract framework (continued)

$$\mathcal{B} \equiv \mathcal{A}\langle\mathcal{F}\rangle := \mathcal{A} \oplus \mathcal{F}, \tag{5}$$

$$(a_1, f_1) \cdot (a_2, f_2) := (a_1 a_2, a_1 f_2 + f_1 a_2 + f_1 f_2) \tag{6}$$

$$\varphi(a+f) := \varphi(a), \qquad \underline{\varphi'(a+f) := \Phi(f)}. \tag{7}$$

Remark

φ models the limit of $\frac{1}{N}\mathrm{Tr}_N$. φ' models the limit of Tr_N but it only captures the perturbation part.

Remark

In the setting of type B (due to Biane, Goodman and Nica 03'), f is considered to be "infinitesimal" and the multiplication was defined by

$$(a_1, f_1) \cdot (a_2, f_2) := (a_1 a_2, a_1 f_2 + f_1 a_2).$$

In this case, one does not need multiplication inside $\mathcal{F}$. ($\mathcal{F}$ is required to be just an $\mathcal{A}$-bimodule.) Then the tuple $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$ was called a **ncps of type B**.

Independences

We define the sets of *alternating sequences*

$$I^{(n)} := \{(i_1, i_2, \dots, i_n) \in I^n \mid i_1 \neq i_2 \neq \cdots \neq i_n\} \ (n \geq 2), \quad I^{(1)} := I$$
$$I^{(\infty)} := \bigcup_{n \in \mathbb{N}} I^{(n)}$$

and sets of tuples of elements

$$\mathcal{A}_{\mathbf{i}} := \mathcal{A}_{i_1} \times \mathcal{A}_{i_2} \times \cdots \times \mathcal{A}_{i_n} \qquad \text{and} \qquad \mathring{\mathcal{A}}_{\mathbf{i}} := \mathring{\mathcal{A}}_{i_1} \times \mathring{\mathcal{A}}_{i_2} \times \cdots \times \mathring{\mathcal{A}}_{i_n}$$

where $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^{(n)}$ and $\mathring{\mathcal{A}}_i := \{a \in \mathcal{A}_i : \varphi(a) = 0\}$.

Definition

A family of subalgebras $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$ containing $1_{\mathcal{A}}$ are said to be *free* if $\varphi(a_1 a_2 \cdots a_n) = 0$ holds for any $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^{(\infty)}$ and any $(a_1, a_2, \dots, a_n) \in \mathring{\mathcal{A}}_{\mathbf{i}}$.

Independences (continued)

$(\mathcal{A}, \varphi)$: unital ncps, $(\mathcal{F}, \Phi)$: $\mathcal{A}$-algebra $\mathcal{F}$ with a linear functional Φ,

Definition (Ben Ghorbal-Schürmann, 2002, terminology was different)

Subalgebras $(\mathcal{F}_i)_{i \in I}$ of $\mathcal{F}$ are said to be **trivially independent** with respect to Φ if $\Phi(f_1 f_2 \cdots f_n) = 0$ holds for every $n \geq 2, \mathbf{i} \in I^{(n)}$ and every $(f_1, f_2, \dots, f_n) \in \mathcal{F}_{\mathbf{i}}$. "All genuine mixed moments vanish".

Definition (Collins, Hasebe, Sakuma 2018)

Let $\mathcal{A}_1$ a subalgebra of $\mathcal{A}$ containing $1_{\mathcal{A}}$ and $\mathcal{F}_1$ a subalgebra of $\mathcal{F}$. The pair $(\mathcal{A}_1, \mathcal{F}_1)$ is said to be **cyclic-antimonotone independent** if

$$\Phi(a_0 f_1 a_1 f_2 \cdots a_{n-1} f_n a_n) = \varphi(a_0 a_n) \left[\prod_{1 \leq i \leq n-1} \varphi(a_i) \right] \Phi(f_1 f_2 \cdots f_n)$$

for $n \in \mathbb{N}$, $a_i \in \mathcal{A}_1$, $f_i \in \mathcal{F}_1$. **Note: in some papers, $\varphi(a_n a_0)$ in RHS**

B'-freeness

Let $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$ be a ncps of type B'. Recall the setting

$$\mathcal{B} := \mathcal{A} \oplus \mathcal{F},$$
$$(a_1, f_1) \cdot (a_2, f_2) := (a_1 a_2, a_1 f_2 + f_1 a_2 + f_1 f_2)$$
$$\varphi(a + f) := \varphi(a), \qquad \varphi'(a + f) := \Phi(f).$$

For a unital subalgebra $\mathcal{A}_1 \subset \mathcal{A}$ and a subalgebra $\mathcal{F}_1 \subset \mathcal{F}$, let $\mathcal{A}_1\langle\mathcal{F}_1\rangle$ be the subalg. of $\mathcal{B}$ generated by $\mathcal{A}_1, \mathcal{F}_1 \subseteq \mathcal{B}$. $\mathcal{A}_1\langle\mathcal{F}_1\rangle$ is the set of linear combinations of elements of the form a_i and $a_1 f_1 a_2 f_2 \cdots a_k f_k a_{k+1}$.

Definition

Let $1 \in \mathcal{A}_i \subset \mathcal{A}$ and $\mathcal{F}_i \subset \mathcal{F}$ (subalg.). We call $(\mathcal{A}_i, \mathcal{F}_i)_{i \in I}$ are **B′-free** if

- (F) $(\mathcal{A}_i)_{i \in I}$ are free with respect to $(\mathcal{A}, \varphi)$,
- (CM) the pair $(\langle \mathcal{A}_i : i \in I\rangle, \langle \mathcal{F}_i : i \in I\rangle)$ of algebras generated by $\{\mathcal{A}_i\}$ and $\{\mathcal{F}_i\}$ is cyclic-antimonotone independent w.r.t. $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$,
- (T) $(\mathcal{F}_i)_{i \in I}$ are trivially independent with respect to $(\mathcal{F}, \Phi)$.

We call $(\mathcal{A}_i, \mathcal{F}_i)_{i\in I}$ are **B′-free** if

(F) $(\mathcal{A}_i)_{i\in I}$ are free with respect to $(\mathcal{A}, \varphi)$,

(CM) $(\langle \mathcal{A}_i : i \in I\rangle, \langle \mathcal{F}_i : i \in I\rangle)$ is cyclic-antimonotone indep. w.r.t. $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$

(T) $(\mathcal{F}_i)_{i\in I}$ are trivially independent with respect to $(\mathcal{F}, \Phi)$.

Theorem (Fujie and H.)

$(\mathcal{A}_i, \mathcal{F}_i)_{i\in I}$ are B'-free if and only if:

1. *the pair $(\mathcal{A}_i, \mathcal{F}_i)$ is cyclic-antimonotone independent for every $i \in I$;*
2. *$(\mathcal{A}_i\langle\mathcal{F}_i\rangle)_{i\in I}$ are infinitesimally free in $(\mathcal{B}, \varphi, \varphi')$, i.e., for every $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^{(\infty)}$ and $(b_1, b_2, \ldots, b_n) \in \mathring{\mathcal{B}}_{\mathbf{i}}$ we have $\varphi(b_1 b_2 \cdots b_n) = 0$ and*

$$\varphi'(b_1 b_2 \cdots b_n) = \begin{cases} \varphi'(b_{(n+1)/2}) \displaystyle\prod_{k=1}^{(n-1)/2} \varphi(b_k b_{n-k+1}) \text{ if } n \text{ is odd and} \\ \qquad i_k = i_{n-k+1} \text{ for all } k = 1, 2, \ldots, (n-1)/2, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Key lemma

Let $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$ be a ncps of type B' and

$$\mathcal{B} \equiv \mathcal{A}\langle \mathcal{F} \rangle := \mathcal{A} \oplus \mathcal{F},$$
$$(a_1, f_1) \cdot (a_2, f_2) := (a_1 a_2, a_1 f_2 + f_1 a_2 + f_1 f_2)$$
$$\varphi(a + f) := \varphi(a), \qquad \varphi'(a + f) := \Phi(f).$$

Let $\kappa_n, \kappa'_n \colon \mathcal{B}^n \to \mathbb{C}$ be free cumulants and infinitesimal free cumulants with respect to (φ, φ'), respectively:

$$\varphi(b_1 b_2 \cdots b_n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} \kappa_{|V|}(b_i : i \in V), \tag{8}$$

$$\varphi'(b_1 b_2 \cdots b_n) = \sum_{\pi \in \mathrm{NC}(n)} \sum_{W \in \pi} \kappa'_{|W|}(b_i : i \in W) \prod_{V \in \pi \setminus \{W\}} \kappa_{|V|}(b_i : i \in V).$$

Rem: κ'_n are recursively obtained by taking formal derivatives of (8), e.g.,

$$\varphi(b) = \kappa_1(b) \quad \Longrightarrow \quad \varphi'(b) = \kappa'_1(b),$$

$$\varphi(b_1 b_2) = \kappa_2(b_1, b_2) + \kappa_1(b_1)\kappa_2(b_2)$$
$$\Longrightarrow \varphi'(b_1 b_2) = \kappa'_2(b_1, b_2) + \kappa'_1(b_1)\kappa_2(b_2) + \kappa_1(b_1)\kappa'_2(b_2)$$

Key lemma (continued)

Let $(\mathcal{A}, \varphi, \mathcal{F}, \Phi)$ be a ncps of type B' and

$$\mathcal{B} \equiv \mathcal{A}\langle\mathcal{F}\rangle := \mathcal{A} \oplus \mathcal{F}, \tag{9}$$

$$(a_1, f_1) \cdot (a_2, f_2) := (a_1 a_2, a_1 f_2 + f_1 a_2 + f_1 f_2) \tag{10}$$

$$\varphi(a+f) := \varphi(a), \qquad \varphi'(a+f) := \Phi(f). \tag{11}$$

Lemma (Characterization of c.a.m.-independence)

Let $(\mathcal{A}_i)_{i\in I}$ be subalgebras of $\mathcal{A}$ containing $1_\mathcal{A}$ and $(\mathcal{F}_i)_{i\in I}$ be subalgebras of $\mathcal{F}$. Then the pair $(\langle \mathcal{A}_i : i \in I\rangle, \langle \mathcal{F}_i : i \in I\rangle)$ is cyclic-antimonotone if and only if the following condition holds.

- *For any $n \in \mathbb{N}$, any indices $i_1, i_2 \dots, i_n \in I$ and elements $a_j + f_j \in \mathcal{A}_{i_j} \oplus \mathcal{F}_{i_j} \subseteq \mathcal{B}$, we have*

$$\kappa'_n(a_1 + f_1, a_2 + f_2, \dots, a_n + f_n) = \Phi(f_1 f_2 \cdots f_n).$$

Proof of the main theorem

Recall that the main theorem asserts:

$$\begin{cases} (\mathcal{A}_i, \mathcal{F}_i)\text{: c.a.m.-indep. } \forall i \in I \\ (\underbrace{\mathcal{A}_i\langle\mathcal{F}_i\rangle}_{=:\mathcal{B}_i})_{i\in I}\text{: i.-free in } (\mathcal{B}, \varphi, \varphi') \end{cases} \iff \begin{cases} (\mathcal{A}_i)_i \text{ is free} \\ (\langle\mathcal{A}_i : i \in I\rangle, \langle\mathcal{F}_i : i \in I\rangle)\text{: c.a.m indep} \\ (\mathcal{F}_i)_{i\in I} \text{ are trivially indep. in } (\mathcal{F}, \Phi) \end{cases}$$

Part $\Longrightarrow$. The trivial indep. is an easy consequence of the definitions. Freeness is a part of i.-freeness.

Now the goal is to prove the c.a.m.-indep.: for any $n \in \mathbb{N}$, any indices $i_1, i_2 \dots, i_n \in I$ and elements $a_j + f_j \in \mathcal{A}_{i_j} \oplus \mathcal{F}_{i_j}$, we have

$$\kappa'_n(a_1 + f_1, a_2 + f_2, \dots, a_n + f_n) = \Phi(f_1 f_2 \cdots f_n). \qquad (*)$$

If some $i_k \neq i_\ell$ then the LHS vanishes by i.-freeness. The RHS also vanishes by trivial indep. So $(*)$ holds.

If $i_1 = i_2 = \cdots = i_n$ then $(*)$ is a consequence of c.a.m.-independence and the key lemma.

Recall that the main theorem asserts:

$$\begin{cases} (\mathcal{A}_i, \mathcal{F}_i)\text{: c.a.m.-indep. } \forall i \in I \\ (\mathcal{A}_i\langle\mathcal{F}_i\rangle)_{i\in I}\text{: i.-free in } (\mathcal{B}, \varphi, \varphi') \end{cases} \iff \begin{cases} (\mathcal{A}_i)_i \text{ is free} \\ (\langle\mathcal{A}_i : i \in I\rangle, \langle\mathcal{F}_i : i \in I\rangle)\text{: c.a.m indep} \\ (\mathcal{F}_i)_{i\in I} \text{ are trivially indep. in } (\mathcal{F}, \Phi) \end{cases}$$

The part $\Longleftarrow$. In case φ, Φ are tracial, the proof is simpler because of:

Lemma (Cébron-Gilliers)

Suppose that φ, φ' are tracial. Then subalgebras $(\mathcal{B}_i)_{i\in I}$ of $\mathcal{B}$ are infinitesimally free if and only if the following condition holds:

1. *For every $n \geq 2$, every $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^{(n)}$ with $i_n \neq i_1$ and $(b_1, b_2, \ldots, b_n) \in \mathring{\mathcal{B}}_{\mathbf{i}}$ we have $\varphi(b_1 b_2 \cdots b_n) = \varphi'(b_1 b_2 \cdots b_n) = 0$.*

- G. Cébron and N. Gilliers, Asymptotic cyclic-conditional freeness of random matrices. arXiv:2207.06249

Recall that the main theorem asserts:

$$\begin{cases}(\mathcal{A}_i,\mathcal{F}_i)\text{: c.a.m.-indep. } \forall i\in I\\ (\mathcal{A}_i\langle\mathcal{F}_i\rangle)_{i\in I}\text{: i.-free in } \ (\mathcal{B},\varphi,\varphi')\end{cases} \Longleftrightarrow \begin{cases}(\mathcal{A}_i)_i \text{ is free}\\ (\langle\mathcal{A}_i : i\in I\rangle,\langle\mathcal{F}_i : i\in I\rangle)\text{: c.a.m indep}\\ (\mathcal{F}_i)_{i\in I} \text{ are trivially indep. in } (\mathcal{F},\Phi)\end{cases}$$

The part $\Longleftarrow$ follows from the following:

Lemma

If $(\mathcal{A}_i)_i$ is free and $(\langle\mathcal{A}_i : i\in I\rangle,\langle\mathcal{F}_i : i\in I\rangle)$ is c.a.m indep., then for every $(i_1,i_2,\ldots,i_n)\in I^{(\infty)}$ with $i_n\neq i_1$ and $B_j=A_j+F_j\in\mathcal{A}_{i_j}\langle\mathcal{F}_{i_j}\rangle$ $=\mathcal{A}_{i_j}\oplus\mathcal{A}_{i_j}\langle\mathcal{F}_{i_j}\rangle_0$ such that $\varphi(B_j)=0$ $(1\leq j\leq n)$, we have

$$\varphi'(B_1B_2\cdots B_n)=\Phi(F_1F_2\cdots F_n) \tag{12}$$

This lemma follows from

$$\varphi'(B_1B_2\cdots B_n)=\underbrace{\varphi'(A_1\cdots A_n)}_{=0}+\sum_{\substack{C_j\in\{A_j,F_j\}\\ 1\leq j\leq n\\ C_j=F_j \text{ for some } j}}\underbrace{\varphi'(C_1C_2\ldots C_n)}_{=\Phi(C_1C_2\ldots C_n)},$$

and the definition of cyclic-antimonotone indep. (Most terms vanish.)

Weak B'-freeness

The other model

$$\{U_i A_i U_i^*, F_i\}_{i \in I} \tag{13}$$

shows a weaker asymptotic independence called **weak B'-freeness**. This is exactly B'-freeness except trivial independence. There is a connection to conditional freeness (due to Bożejko, Speicher and Leinert).

Theorem

Let $(\mathcal{B}_i = \mathcal{A}_i \langle \mathcal{F}_i \rangle)_{i \in I}$. Let P be an element of $\mathcal{F}$ with $\Phi(P) \neq 0$. Assume that $(\mathcal{B}_i)_{i \in I \sqcup \{p\}}$ is weakly B'-free, where $\mathcal{B}_p := \mathbb{C}1_{\mathcal{A}} \oplus \langle P \rangle$. We define a linear functional $\varphi_P \colon \mathcal{B} \to \mathbb{C}$ as follows:

$$\varphi_P(b) := \frac{\Phi(Pb)}{\Phi(P)}, \qquad b \in \mathcal{B}.$$

Then $(\mathcal{B}_i)_{i \in I}$ are conditionally free with respect to (φ, φ_P) if and only if $(\mathcal{F}_i)_{i \in I}$ are boolean independent with respect to φ_P.

Asymptotic infinitesimal freeness of principal minors

Let $P = \mathrm{diag}(1, 1, \dots, 1, 0) \in \mathrm{M}_N(\mathbb{C})$, $Q := I - P = \mathrm{diag}(1, 0, 0, \dots, 0)$,

$$\tilde{A}_i := PU_iA_iU_i^*P.$$

The family

$$\{(U_iA_iU_i^*, Q)\}_i \tag{14}$$

is asymptotically weakly B'-free. Let $\{(a_i, q)\}_i$ be the limiting nc r.v.s for (14) in $(\mathcal{B}, \varphi, \varphi')$, where $\mathcal{B} = \mathcal{A} \oplus \mathcal{F}$ as before. ($a_i \in \mathcal{A}, q \in \mathcal{F}$).

1. The limits of principal minors can be interpreted as $\tilde{a}_i := pa_ip$, where $p := 1_{\mathcal{B}} - q$.
2. Using the fact that $(\{a_i\}, q)$ is c.a.m.indep., it is easy to check (but important) that p is infinitesimally free from $\mathcal{A}$ in $(\mathcal{B}, \varphi, \varphi')$.

So, we can apply Févier and Nica's work on infinitesimal free compressions to pa_ip.

- M. Février and A. Nica, Infinitesimal non-crossing cumulants and free probability of type B, J. Funct. Anal. 258 (2010), no. 9, 2983–3023.

Asymptotic infinitesimal freeness of principal minors (2)

Following Février-Nica, let $\mathcal{B}_p := p\mathcal{B}p \subseteq \mathcal{B}$ with unit p and ψ, $\psi' : \mathcal{B}_p \to \mathbb{C}$:

$$\psi = \varphi, \qquad \psi' = \varphi + \varphi'.$$

Note that $\psi'(p) = \varphi(p) + \varphi'(p) = 1 + \varphi'(1_{\mathcal{B}} - q) = 1 - \varphi'(q) = 1 - 1 = 0$.

Lemma (An easy consequence of Février-Nica 2010)

Let $(\mathcal{A}_i)_{i\in I}$ be free subalgebras in $(\mathcal{A}, \varphi)$. Then $(p\mathcal{A}_i p)_{i\in I}$ are infinitesimally free in $(\mathcal{B}_p, \psi, \psi')$.

Theorem (Fujie-H.)

Consider the decomposition of $\tilde{A}_i = PU_iA_iU_i^*P$:

$$\tilde{A}_i = U_iA_iU_i^* \oplus (PU_iA_iU_i^*P - U_iA_iU_i^*) \in \mathrm{M}_N(\mathbb{C}) \oplus \mathrm{M}_N(\mathbb{C}),$$

$P = 1 \oplus (-Q)$, *and let* $\psi_N := \mathrm{tr}_N \oplus 0$ *and* $\psi'_N := \mathrm{tr}_N \oplus \mathrm{Tr}_N$. *Then* $\{\tilde{A}_i\}_{i\in I}$ *is asympt. infinitesimally free a.s. in* $(P(\mathrm{M}_N \oplus \mathrm{M}_N)P, \psi_N, \psi'_N)$.

The crucial idea is to separate the main part and perturbation part.

Multivariate inverse Markov-Krein transform

Let $\underline{\kappa}'_n \colon \mathcal{B}_p^n \to \mathbb{C}$ be the n-th infinitesimal free cumulant with respect to $(\psi, \psi') = (\varphi, \varphi + \varphi')$. A formula of Février-Nica yields (with $\tilde{a}_i := p a_i p$)

$$\underline{\kappa}'_n(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (1-n)\kappa_n^{\varphi}(a_1, a_2, \ldots, a_n). \tag{15}$$

Note: Here we do not require a_i's to be free. We only need p is infinitesimally free from $\{a_i\}$.

By the moment-cumulant formula

$$\begin{aligned}\psi'(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_n) &= \sum_{\pi \in \mathrm{NC}(n)} \underline{\kappa}'_\pi[\tilde{a}_1, \tilde{a}_2, \cdots \tilde{a}_n] \\ &= \sum_{\pi \in \mathrm{NC}(n)} (|\pi| - n)\kappa_\pi^{\varphi}[a_1, a_2, \ldots, a_n]\end{aligned} \tag{16}$$

and hence, by the moment-cumulant formula

$$\begin{aligned}\varphi'(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_n) &= \psi'(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_n) - \varphi(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_n) \\ &= \sum_{\pi \in \mathrm{NC}(n)} (|\pi| - n - 1)\kappa_\pi^{\varphi}[a_1, a_2, \ldots, a_n]\end{aligned}$$

Multivariate inverse Markov-Krein transform

Considering $\#\,\mathrm{Kr}(\pi) = -|\pi| + n + 1$, we can write

$$-\varphi'(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_n) = \sum_{\pi\in\mathrm{NC}(n)} \#\,\mathrm{Kr}(\pi)\kappa_\pi^\varphi[a_1, a_2, \ldots, a_n] \qquad (17)$$

When $a_1 = a_2 = \cdots = a_n$, the RHS is exactly the n-th moment with respect to the Rayleigh measure (inverse Markov-Krein transform of the distribution of a_i with respect to φ) (Fujie-Hasebe 2022). So (17) is worth being called a multivariate (inverse) Markov-Krein transform, cf. Arizmendi, Cébron and Gilliers.

- O. Arizmendi, G. Cébron and N. Gilliers, Combinatorics of cyclic-conditional freeness. Arxiv:2311.13178
- K. Fujie and T. Hasebe, The spectra of principal submatrices in rotationally invariant Hermitian random matrices and the Markov–Krein correspondence, ALEA Lat. Am. J. Probab. Math. Stat. 19 (2022), no. 1, 109–123.

Markov-Krein transform and infinitesimal free convolution

The previous formula implies:

Proposition

Let $a \in \mathcal{A}$. Let μ be the distribution of a with respect to φ and τ the inverse Markov–Krein transform of μ. Then the infinitesimal distribution of $\tilde{a} := pap$ with respect to $(\mathcal{B}_p, \psi, \psi')$ is $(\mu,\, \mu - \tau)$.

Recall that if $(\mathcal{A}_i)_{i \in I}$ is free in $(\mathcal{A}, \varphi)$ then $(p\mathcal{A}_i p)_{i \in I}$ are inf. free in $(\mathcal{B}_p, \psi, \psi')$ (F-N). From the obvious identity $p(a_1 + a_2)p = pa_1p + pa_2p$:

Corollary

Let $\mu := \mu_1 \boxplus \mu_2$ be the free convolution of distributions μ_1, μ_2. Let τ_i, τ be the inverse Markov–Krein transforms of μ_i, μ ($i = 1, 2$), resp. Then

$$\underbrace{(\mu_1, \mu_1 - \tau_1) \boxplus_{inf} (\mu_2, \mu_2 - \tau_2)}_{\textit{infinitesimal free convolution}} = (\mu, \mu - \tau).$$

A similar formula holds for inf. mult. free convolution.

The reasoning is different but the multiplicative analogue also holds:

Proposition

Let $\mu := \mu_1 \boxtimes \mu_2$ *be the multiplicative free convolution of distributions* μ_1, μ_2*. Let* τ_i, τ *be the inverse Markov–Krein transforms of* μ_i, μ $(i = 1, 2)$*, resp. Then*

$$(\mu_1, \mu_1 - \tau_1) \boxtimes_{inf} (\mu_2, \mu_2 - \tau_2) = (\mu, \mu - \tau).$$

A similar formula holds for inf. mult. free convolution.

Thank you!