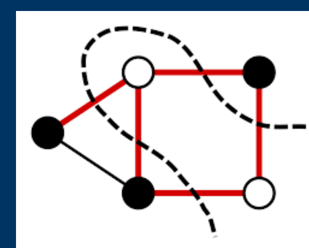
Free Probability and Noncommutative Optimization

Maximize $< \phi | A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 | \phi >$



Max-Cut

Hamoon Mousavi (Simons Institute at UC Berkeley)

Eric Culf (University of Waterloo), Taro Spirig (University of Copenhagen)

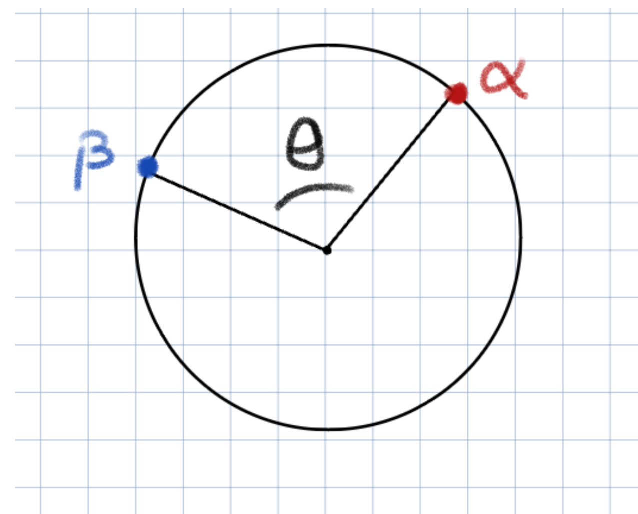
We talk about

- Noncommutative Constraint Satisfaction Problems
- Distribution of eigenvalues of pairs of random unitaries
- Free probability for understanding this distribution on eigenvalues

Relative Distribution

Relative Phase

between eigenvalues of random unitaries



Random eigenvalues of pair of unitaries

Eigenvalues of one unitary

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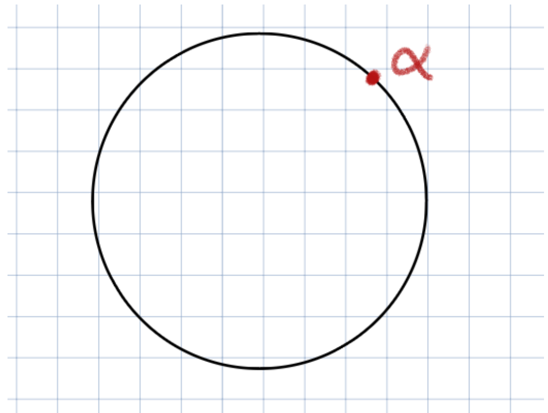

The random eigenvalue on the unit circle

Eigenvalues of one unitary

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Eigenvalues of one unitary

- Sample a Haar random unitary X
- Sample an eigenvalue α
    - Let P_α be the projection onto α-eigenspace
    - Sample α with probability $\mathrm{tr}(P_\alpha)$

Eigenvalues of pairs of unitaries

(and their relative phase)

- Sample Haar random unitaries X and Y

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- Sample an eigenvalue α of X
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  - With probability $\mathrm{tr}(P_\alpha Q_\beta) = < P_\alpha, Q_\beta >$
  - P_α projection onto α-eigenspace of X
  - Q_β projection onto β-eigenspace of Y

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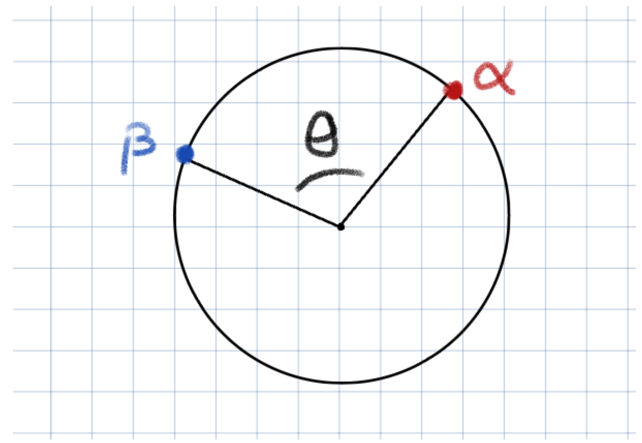

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- Sample eigenvalue (α, β) of X and Y
- We are interested in the random variable

$$\theta = \angle\alpha^*\beta = \angle\beta - \angle\alpha$$

Relative Distribution

Given a distribution on pairs of unitaries (X, Y) we can study the "distribution of the relative phase θ"

Fixed inner product distribution on pairs of unitaries

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  - Sample a Haar random unitary U
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- Clearly $< X, Y > = < A, B >$

Relative Distribution of A and B

- Sample a Haar random unitary U
- Let $X = UA$ and $Y = UB$
- Sample an eigenvalue α of X and β of Y
    - With probability $\mathrm{tr}(P_\alpha Q_\beta)$
- Return $\theta = \angle \alpha^* \beta$

An example

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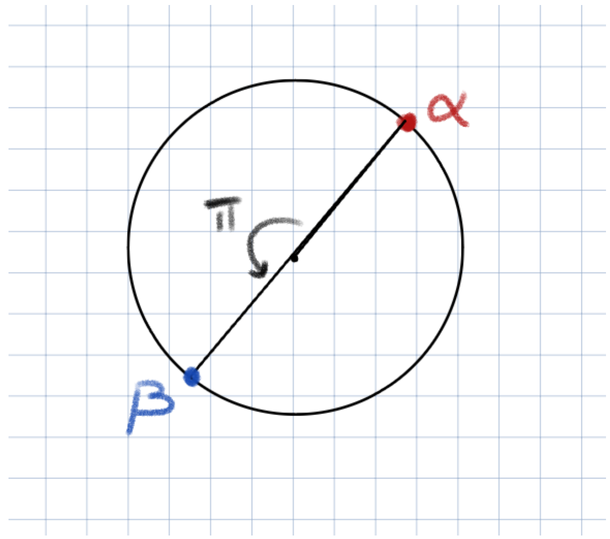
An example

- Suppose $< A, B > = -1$ then $B = -A$
- Then for any sample (X, Y) we also have $Y = -X$
    - so if (α, β) is a sample from our distribution of eigenvalues

      with probability one we have $\beta = -\alpha$

An example



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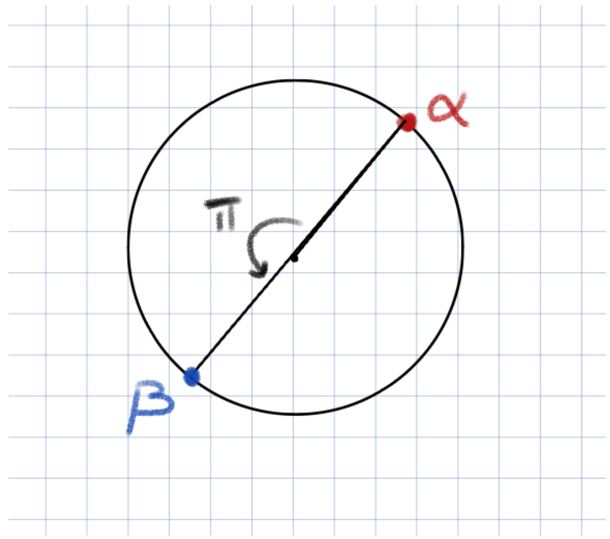

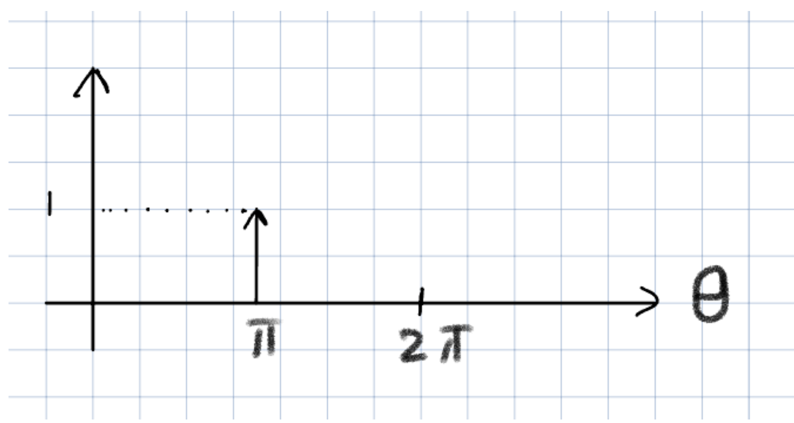

PDF of the relative distribution is the Dirac delta at π

Typical behaviour (informal)

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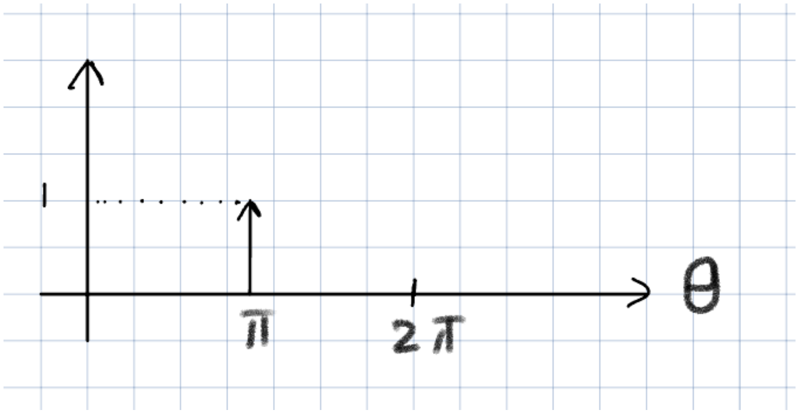
- Let $\lambda = < A, B >$ and let θ be the relative phase r.v.
- Then $\mathbb{E}e^{i\theta} = \lambda$
- So we expect θ to be somewhere around $\angle\lambda$

- In our example $\lambda = -1$ and $\angle\lambda = \pi$



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Cauchy Law (this paper)

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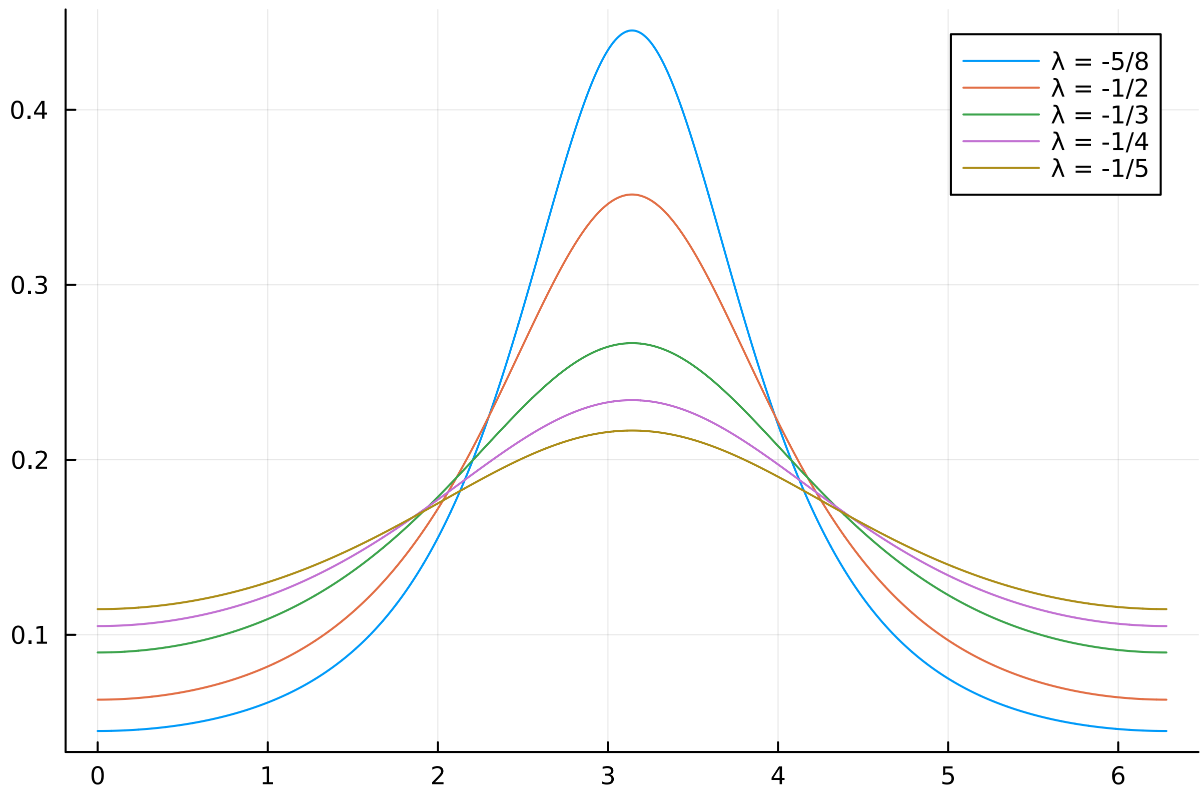
Relative distribution of A and B is a wrapped Cauchy distribution with parameters that only depends on $< A, B >$

- This holds only as $\dim \to \infty$
- If $\lambda = < A, B >$ then
  - The peak position is at $\angle\lambda$
  - The scale factor is $\ln(1/|\lambda|)$



PDF of relative distribution of A and B for various values of $\lambda = < A, B >$

Proof Idea

- Let $\Delta_{A,B} : \mathscr{B}([0,2\pi)) \to \mathbb{R}_{\geq 0}$ denote the distribution function of the relative distribution of A and B

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- Then $\Delta_{A,B}(E) = \mathbb{E}_U(w_{UA,UB}(E))$

- Here $w_{UA,UB}(E)$ is the sum of $\mathrm{tr}(P_\alpha Q_\beta)$ whenever $\angle \alpha^* \beta \in E$

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part 1: convergence in distribution

- If we let

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our goal is to show that the $\Delta_{A,B}$ converges to the wrapped Cauchy distribution with parameter λ

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- We just need to show that the characteristic function of $\Delta_{A,B}$

$$\chi_{\Delta_{A,B}}(n) \to \lambda^n$$

Proof Idea
part 2: characteristic function

- The characteristic function of relative distribution $\Delta_{A,B}$ is

$$\chi_{\Delta_{A,B}}(n) = \int \mathrm{tr}[U^{-n}(UD)^n]dU$$

where D is A^*B

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- The characteristic function of relative distribution $\Delta_{A,B}$ is

$$\chi_{\Delta_{A,B}}(n) = \int \mathrm{tr}[U^{-n}(UD)^n]dU$$

where D is A^*B

- In fact we can assume D is diagonal

Proof Idea
part 3: free independence

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- Goal $\int \mathrm{tr}[U^{-n}(UD)^n]dU \quad \to \quad \mathrm{tr}(D)^n$

- For $n = 1,2$, it is easy to show that the integral is exactly $\mathrm{tr}(D)^n$

- As dimension grows, we have $U, D \to u, d$ in *-distribution where u and d are free

Proof Idea
part 3: free independence

- So $$\int \mathrm{tr}[U^{-n}(UD)^n]dU \quad \to \quad \tau(u^{-n}(ud)^n)$$

  and the right hand side is just $\tau(d)^n$

  which is $\mathrm{tr}(D)^n$

Application to Optimization

Discrete optimization (Example)

$$\max \sum w_{ij} x_i x_j$$

s.t. $\quad x_i^2 = 1$

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$$\max \sum w_{ij} x_i x_j$$

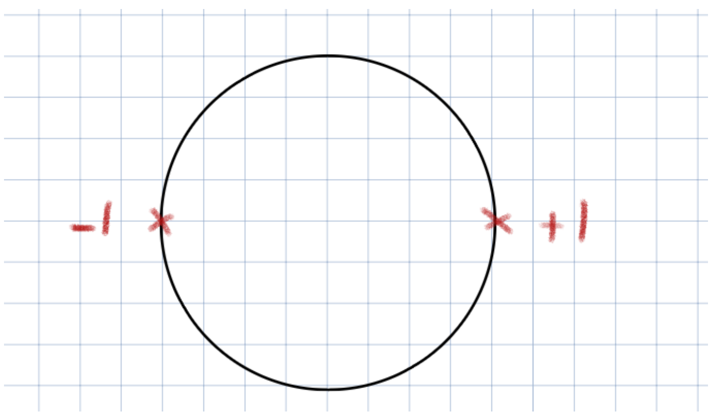
$$\text{s.t.} \quad x_i^2 = 1$$

- Relax to $x_i \in [-1,1]$ and solve

- Round to nearest discrete point $\{-1,1\}$

Operator Optimization

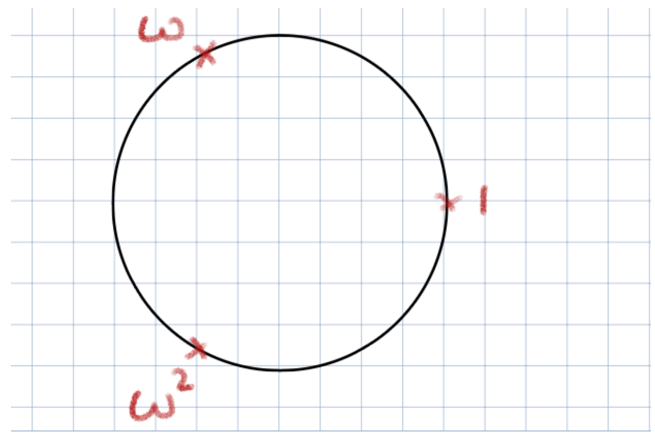
- Imagine optimizing over unitaries with discrete set of eigenvalues



$A^*A = 1$ **and** $A^2 = 1$

Operator Optimization

- Imagine optimizing over unitaries with discrete set of eigenvalues



$A^*A = 1$ **and** $A^3 = 1$

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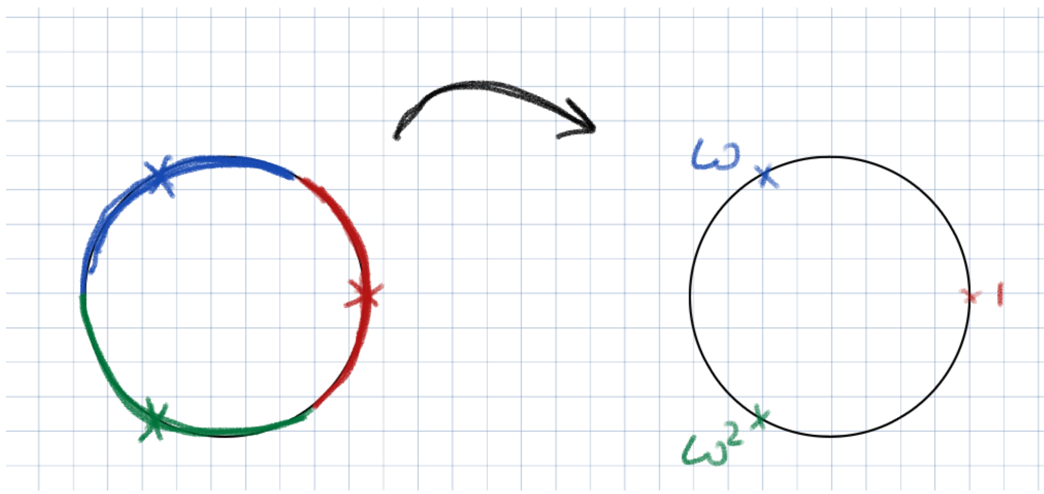

Operator Optimization

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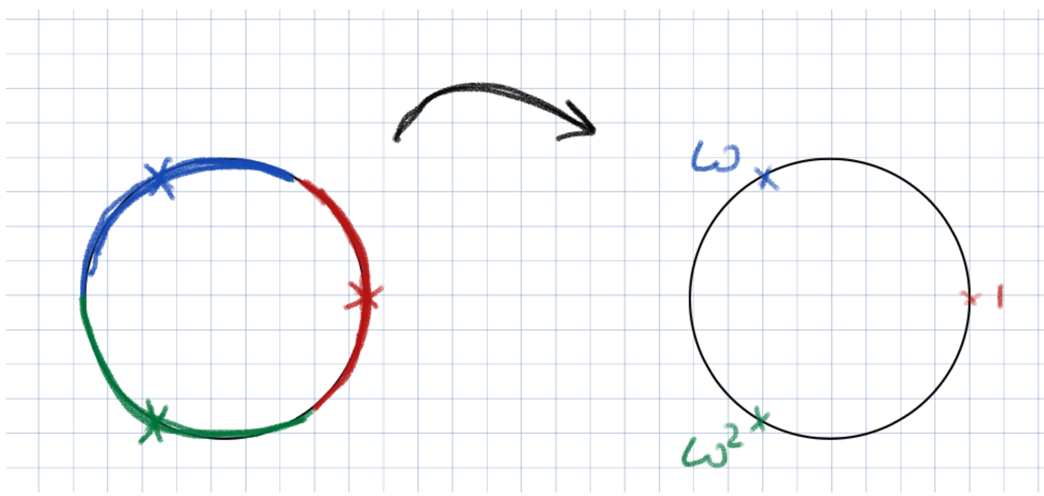

- $\tilde{A}$ is the nearest discrete unitary (of order-3) to A

$$\max \sum w_{ij} < A_i, A_j >$$

s.t. A_i are some discrete unitaries

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- Suppose we are given a solution $A_1, \ldots, A_n$ to the relaxation
- Premultiplying by a Haar unitary U does not change the value

$$UA_1, \ldots, UA_n$$

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- Suppose we are given a solution $A_1, \ldots, A_n$ to the relaxation
- Premultiplying by a Haar unitary U does not change the value

$$UA_1, \ldots, UA_n$$

- How good is the following solution in expectation?

$$\tilde{UA}_1, \ldots, \tilde{UA}_n$$

- We want to compare

$$\sum w_{ij} < A_i, A_j >$$

and

$$\mathbb{E}_U \sum w_{ij} < \tilde{U} A_i, \tilde{U} A_j >$$

- We want to compare

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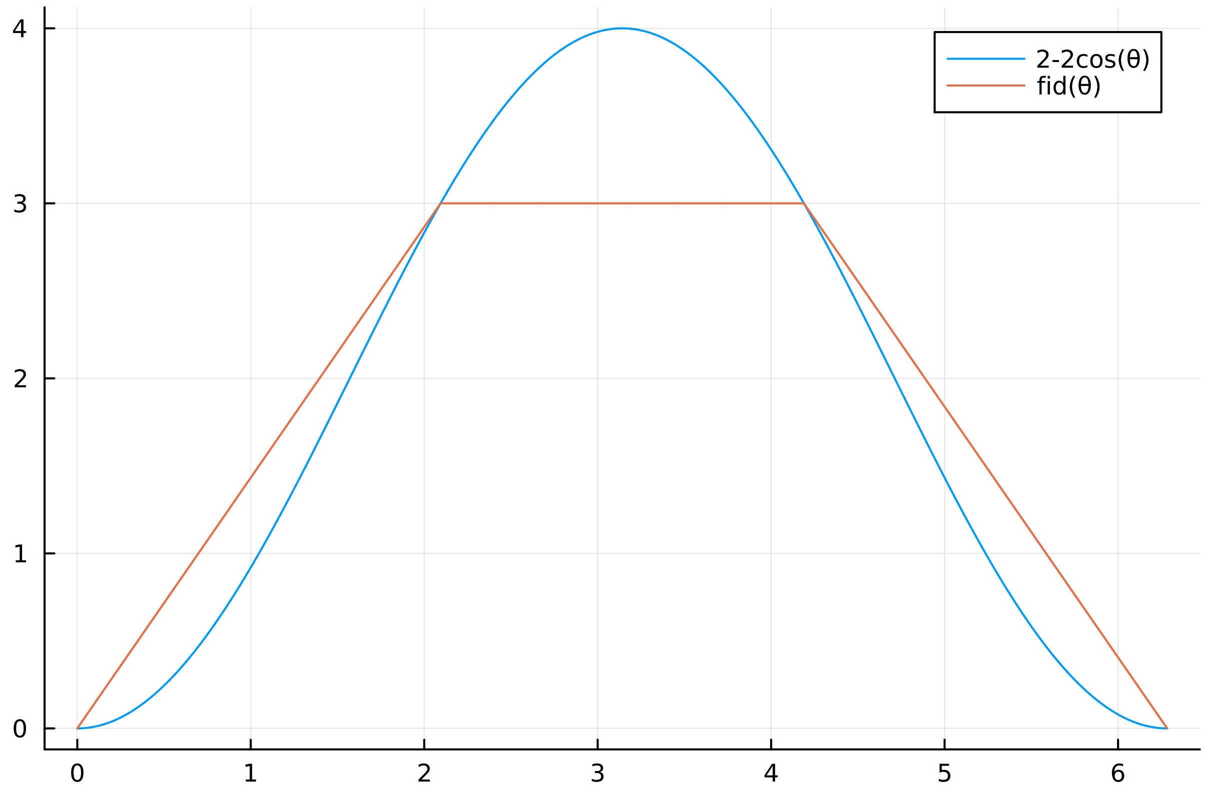
and

$$\mathbb{E}_U < \tilde{U}A, \tilde{U}B > = \int \mathrm{fid}(\theta) d\Delta_{A,B}(\theta)$$

What does fidelity look like?



Theorem

$$\mathbb{E}_U(1 - < \tilde{U}A, \tilde{U}B >) \geq 0.864(1 - < A, B >)$$

Application of Cauchy Law to Optimization

If there is a unitary solution, we know there exists a nearby discrete unitary solution

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If there is a unitary solution, we know there exists a nearby discrete unitary solution

You can often find a unitary solution by solving an SDP

- In fact we can calculate

$$\mathbb{E}_U \mathrm{tr}(P(\tilde{U}A, \tilde{U}B)) = \int \mathrm{fid}_P(\theta) d\Delta_{A,B}(\theta)$$

where fid_P is independent of A and B!

Noncommutative Constraint Satisfaction Problems

Example of a constraint satisfaction problem

Magic Square

$x_{ij} \in \{+1, -1\}$

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

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A contradiction

Example of a constraint satisfaction problem

Magic Square with Matrices

$$X_{ij}^* X_{ij} = I$$

$$X_{ij}^* = X_{ij}$$

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Row product $+I$

Column product $-I$

No longer a contradiction

Example of a constraint satisfaction problem

Matrix solution

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
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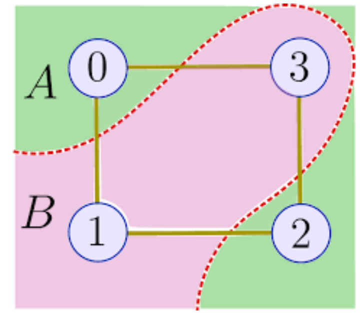
Optimizing CSPs

Max-Cut



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- Variables: $x_1, \ldots, x_n \in \{-1, +1\}$
- Constraints: $x_i \neq x_j$ for every edge
- Goal: Maximize number of constraints satisfied

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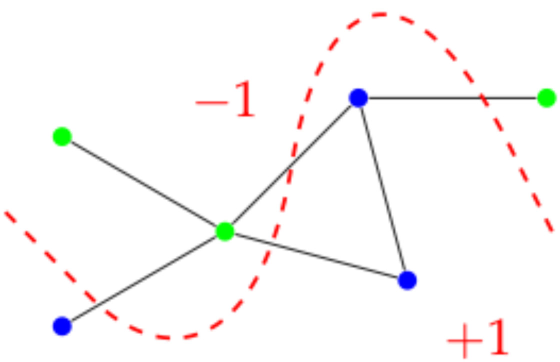


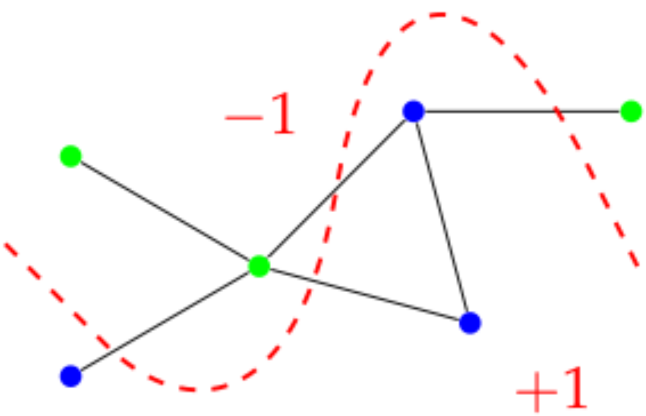
(a) Example of a cut

$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

(b) Max-Cut as a polynomial optimization

Max-Cut



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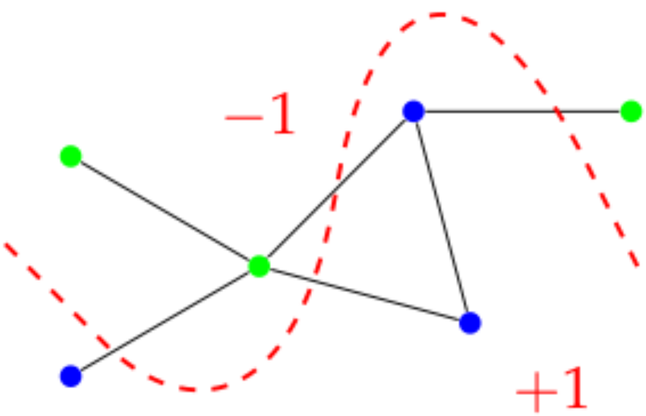

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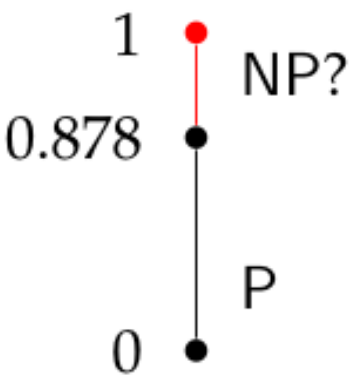
Noncommutative Max-Cut

$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - \langle X_i, X_j \rangle}{2}$$

$$\text{subject to:} \quad X_i \text{ unitary with eigenvalues } \pm 1.$$

Transition in Complexity (Max-Cut)



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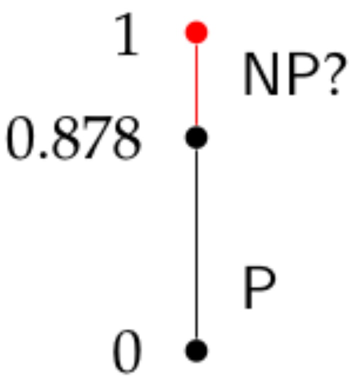
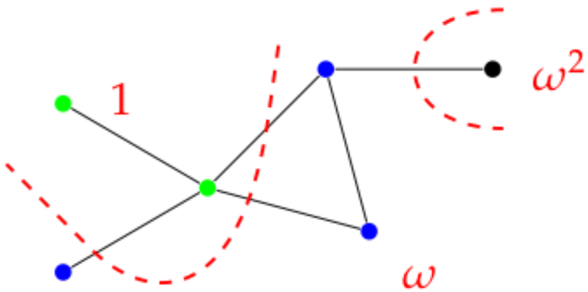


(a) Max-Cut

(b) Noncommutative Max-Cut

Max-3-Cut



(a) Example of a partition of vertices into three subsets

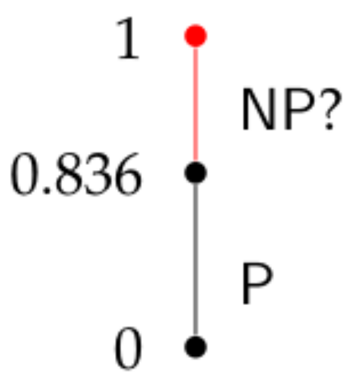
$$\begin{aligned} \text{maximize:} \quad & \sum_{(i,j)\in E} \frac{2 - x_i^* x_j - x_j^* x_i}{3} \\ \text{subject to:} \quad & x_i \in \{1, \omega, \omega^2\}, \end{aligned}$$

(b) Max-3-Cut as a polynomial optimization

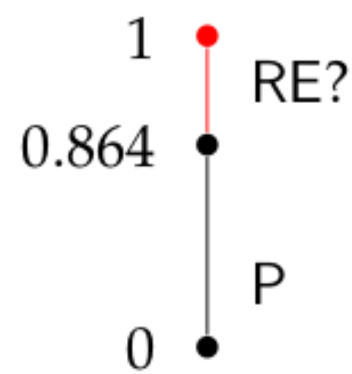
Noncommutative Max-3-Cut

$$\begin{aligned} \text{maximize:} \quad & \sum_{(i,j)\in E} \frac{2 - \langle X_i, X_j \rangle - \langle X_j, X_i \rangle}{3} \\ \text{subject to:} \quad & X_i \text{ unitary with eigenvalues } 1, \omega, \omega^2. \end{aligned}$$

Transition in Complexity (Max-3-Cut)

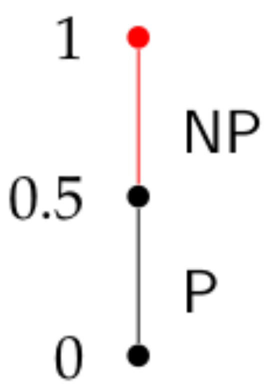


(a) Max-3-Cut

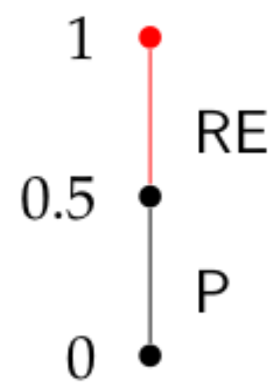


(b) Noncommutative Max-3-Cut

Transition in Complexity (3-XOR)



(a) 3-XOR



(b) Noncommutative 3-XOR

Summary

- Free Probability => Algorithmic Results for NC-CSPs
- Many open problems: Max-4-Cut, Unique-Games, Grothendieck Inequalities, ...
- Hardness: Noncommutative PCP, Noncommutative UGC, ...